Description of the Session 2: Understanding and using representations of fractions

In this session, participants will first have a conversation from a CCA from the last session focused on the meaning of "mathematical knowledge for teaching." Then participants will discuss the important role of representations in mathematics and in mathematics teaching. They will then explore representations of a particular fraction (¾) and examine explanations for the different representations of ¾. Work with these representations of ¾ will continue in Session 3. Near the end of the session, participants will be introduced to different types of connections that can be made using representations: between student thinking and a representation, within representations of the same type, across representations of the same type, across representations of different types, and between a representation and the problem statement.

Activities and goals of the session

Activities	Times	Corresponding parts of the session	Goals
Conversation about a CCA from the last session	10 minutes		• Participants will understand the meaning of "mathematical knowledge for teaching."
I. Preview	5 minutes	Part 1	• Participants will be oriented to the work of the session.
II. Introduction to the importance of representation in mathematics and mathematics teaching	10 minutes	Part 2	• Participants will be able to identify ways in which representations are foundational in mathematics and in mathematics teaching.
III. Exploring and explaining representations of 3/4	70 minutes	Parts 3, 4 & 5	• Participants will be able to: ○ interpret and explain multiple representations of a fraction; ○ use the concepts of the whole and equal parts to explain and critique explanations of representations of fractions. ○ understand diverse explanations for a fraction representation; and ○ compare and critique explanations for a fraction representation.
IV. Types of connections to be made using representations	20 minutes	Part 6	• Participants will develop an awareness of different types of connections that can be made with representations.
IV. Wrap up	5 minutes	Part 7	• Participants will understand ways of connecting the session content to their classroom.

Classroom Connection Activities

Required	Optional
Type of task: Practice and extension of in-class work Description: Creation and analysis of representations of 4/3 Type of task: Describing practice Description: Make notes about how representations are connected in the classroom Type of task: Analysis of curriculum materials Description: Identify representations used to support the learning of fractions and comparing fraction in textbooks.	Type of task: Reading Description: Watanabe (2002) article on tools and methods for representing fractions, fraction notation, and language

Preparing for the session

☐ Make copies as needed:

- *Resources:* Handout: Representations of 3/4 (Part 3); Transcript: Comparing four-fourths and four-eighths (Part 6); Transcript: One-eighth of twenty-four (Part 6); Transcript: Representing one-fourth and one-half (Part 6); Transcript: Representing the shaded part of the rectangle (Part 6)
- *Supplements:* Reading: Math matters (Part 2); Reading: Principles and standards for school mathematics (Part 2); Handout: Large versions of images (Part 4); Math notes: Analysis of representations of 3/4 (Part 4); Handout: Connection in the session: Between participant thinking and a representation (Part 6)

☐ Customize the Classroom Connection Activities and make copies as needed

☐ Test technical setups (internet connection, speaker, projector)

Developing a culture for professional work on mathematics teaching (ongoing work of the facilitator throughout the module)

1. Encourage participation: talking in whole-group discussions; rehearsing teaching practices; coming up to the board as appropriate.
2. Develop habits of speaking and listening: speaking so that others can hear; responding to others' ideas, statements, questions, and teaching practices.
3. Develop norms for talking about teaching practice: close and detailed talk about the practice of teaching; supporting claims with specific examples and evidence; curiosity and interest in other people's thinking; serious engagement with problems of mathematics learning and teaching.
4. Develop norms for mathematical work:
    a) Reasoning: explaining in detail; probing reasons, ideas, and justifications; expectation that justification is part of the work; attending to others' ideas with interest and respect.
    b) Representing: building correspondences and making sense of representations, as well as the ways others construct and explain them.
    c) Carefully using mathematical language.
5. Help participants make connections among module content and develop the sense that this module will be useful in helping them improve their mathematics teaching, their knowledge of mathematics, their understanding of student thinking, and their ability to learning from their own teaching.
6. Help participants understand connections between module content and the Common Core Standards for School Mathematics.

*Scope of the module (focal content of this session in **bold**)*

Mathematics	Student thinking	Teaching practice	Learning from practice
• **representing fractions** • defining fractions • using and explaining methods and representations for comparing fractions • understanding how equivalence (of fractions) can be represented and used	• identifying and analyzing student conceptions, explanations, and representations of fractions • identifying and analyzing student strategies for comparing fractions	• **selecting and generating representations** • **connecting representations** • narrating the process of representing • supporting students in narrating the use of a representation • recording contributions and emerging mathematical ideas	• studying public recording space to learn from practice • using a conceptual framework to guide the planning, use, and analysis of public recording space

Conversation about a Classroom Connection Activity from last session (~10 minutes)

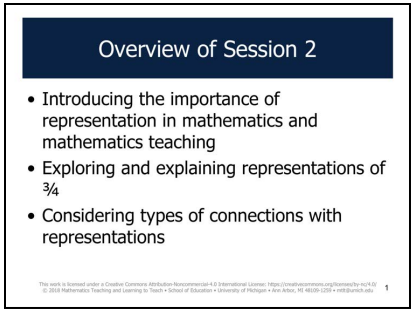
Goals	***Instructional sequence***	***Resources***
• Participants will understand that each session will start with an uptake of work done between sessions. • Participants will appreciate the role of CCAs in the module. • Participants will understand the meaning of "mathematical knowledge for teaching."	1. Remind participants about the important role that the CCAs play in the module. 2. Discuss the mathematical knowledge for teaching article.	• Ball, Hill, and Bass (2005) article from Session 1 Classroom Connection Activities

Detailed description of activity	Comments & other resources
1. Introduce the routine of beginning each session with a discussion of a CCA: • Explain that each session will start with a 10-minute discussion of ideas or examples from the Classroom Connection Activities completed since the previous session. • Emphasize that it is important for participants to bring ideas and materials related to the CCAs to the sessions because these ideas and materials are what make this portion of the session so valuable. • Explain that, for this session, participants will focus on the "Knowing Mathematics for Teaching" article they read as part of the CCA.	*Discussing CCA work from previous sessions establishes its importance in the module.* *Note that most of the time you will be taking up items from the CCAs that are more practice based, so it is unusual to be discussing a reading. However, we have chosen to focus on a reading in this case because it helps to unpack the kind of mathematics that is foundational to the module.*
2. Discuss the Ball, Hill, and Bass article on mathematical knowledge for teaching with participants. Ask participants: • In your own words, what is mathematical knowledge for teaching? Why is it important? • Was there anything mentioned as an element of mathematical knowledge for teaching that was surprising to you? • As you read, did you think of any additional examples of mathematical knowledge for teaching that were not included in the article?	*Mathematical knowledge for teaching (MKT) is a professional knowledge of mathematics that is different from that demanded by other mathematically intensive occupations. MKT is deep and detailed knowledge of mathematics and mathematical reasoning that support the predictable and recurrent tasks that teachers face (beyond procedural fluency, focusing on tasks like error analysis, careful use of mathematical language).* *MKT is important because it is at the core of the daily work of mathematics teaching. It is also important because it has been shown to have a demonstrable effect on student learning.* *Participants may come up with multiple additional examples of MKT. Listen to see if they indicate knowledge that is special to the work that teachers do. For instance, knowing how to use technology in doing mathematics may not be an example, whereas knowing what particular solutions indicate about the ways that students may have used technology may be an example.*

Part 1: Preview (~5 minutes)

Goals	***Instructional sequence***	***Resources***
• Participants will be oriented to the work of the session.	1. Introduce Session 2 and watch the video.	• Video (03:37): Introduction to session

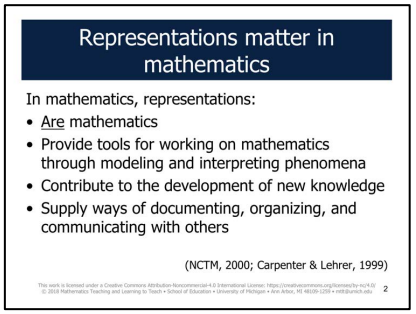
Detailed description of activity	Comments & other resources
1. Introduce the session: Session 1 focused on seeing the mathematical work of teaching, interpreting student thinking, and considering teaching practices. This session focuses on: • Introducing the importance of representation in mathematics and in mathematics teaching • Exploring and explaining representations of 3/4 • Considering types of connections with representations Have participants watch the *video* in which Dr. Ball presents an overview of the content that will be covered in Session 2.	

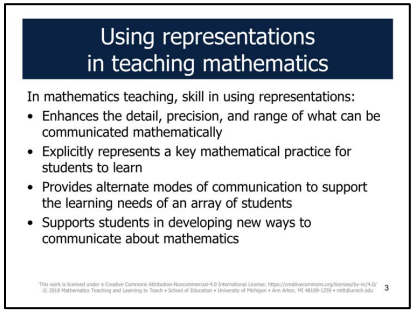


Part 2: The importance of representations in mathematics and mathematics teaching (~10 minutes)

Goals	***Instructional sequence***	***Resources***
• Participants will be able to identify ways in which representations are foundational in mathematics and in mathematics teaching.	1. Introduce Part 2 and watch the video with slides. 2. Have participants discuss (or record) ideas about how representations are foundational to the teaching and learning of fractions.	• Video (05:16): Importance of representations

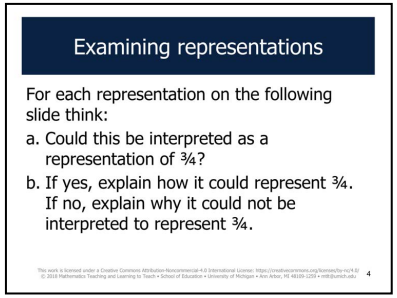
Detailed description of activity	Comments & other resources
1. Introduce Part 2: Representations are a central part of the mathematics and the teaching practices in this module. This part of the session builds the case for a focus on representations in professional learning for mathematics teachers. Have participants watch the *video* in which Dr. Ball describes how representations are foundational to both mathematics and mathematics teaching. Ask participants to discuss one or more of the following questions: • What representations did you use in your mathematics teaching today? • How did you use them and what purposes did you have for using them? • Are there ideas that occurred to you about representations in mathematics and teaching as you watched the video?	*Tip: When showing the video, initially have both the slides and video visible. As the video progresses, switch to a full screen video of the slides and allow the video to continue playing in the background.* *Participants might generate ideas such as:* • *If students can explain a representation, their understanding is greater than if they can just perform a computation/manipulation of a representation.* • *Teachers need to know multiple representations to meet the needs of all students.* • *Teachers need to be able to understand and assess invented representation.* • *Teachers need to be able to assess representations to figure out when to use and when to not use.*
2. Ask participants to discuss (or record) any ideas they have about how representations are foundational to the teaching and learning of fractions.	



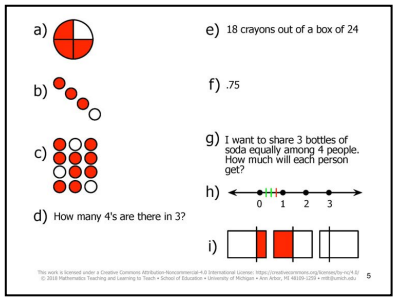


Part 3: Analyzing representations of ¾ (~20 minutes)

Goals	***Instructional sequence***	***Resources***
• Participants will be able to interpret and explain multiple representations of a fraction.	1. Introduce Part 3 and watch the video with slides. 2. Participants record their thinking about each representation.	• Video (01:49): Task introduction • Handout: Representations of ¾

Detailed description of activity	Comments & other resources
1. Introduce Part 3 by having participants watch the *video* in which Dr. Ball introduces the task that participants will be working on in this segment – looking at potential representations of ¾ and deciding: • Could this be interpreted as a representation of ¾? • If yes, explain how it could represent ¾. If no, explain why it could not be interpreted to represent ¾. 	*Participants may move to talking about teaching practices when answering the first question. Encourage participants to focus on mathematics in this part of the session.* *CCSSM Link: In working on this activity, participants are engaged in several of the CCSSM mathematical practices including: making sense of problems and persevering in solving them (#1), reasoning abstractly and quantitatively (#2), constructing viable arguments and critiquing the reasoning of others (#3), and precisely communicating their thinking to others (#6). While the major focus on fractions starts in Grade 3, in the earlier grades, students work on foundational ideas such as partitioning shapes into equal shares and naming a share using fraction language (1.G.3, 2.G.3, 3.G.2).* *See Session 2, Part 4 for a document explaining how each representation can be interpreted as a representation of ¾. You may want to consider distributing this document to participants at the end of the session.* *See Session 3, Part 6 for the "Working Definition of a Fraction." While the participants will not see this working definition until Session 3, the work that they are doing in Session 2 is designed to begin to elicit aspects of the definition. Reviewing this working definition now may be useful for you in preparing for Session 2.*

2. Distribute the *Handout: Representations of ¾.* Have participants work independently on deciding whether each figure could be interpreted as a representation of ¾. Then provide time for participants to discuss their thinking with a partner.



Be sure to point out that you are not asking participants to consider whether they would use the representation with their students. Asking about a representation's difficulty for students and how you could use it in the classroom are important questions for teaching, but in order to answer such questions, teachers first need to be able to analyze a representation and explain how it does or does not represent the intended idea. It is that careful analysis and explanation that this task is meant to address; it is not about whether and how to use the representations with students.

Allowing initial time to independently consider the representations will support participant interactions later on. In the next segment, participants will have the opportunity to discuss particular points about several of the representations.

Participants may comment that knowing multiple representations is important for teaching because different students understand different representations. Try to help participants see that there is a larger issue – different representations make different mathematics visible.

Part 4: Examining explanations for representations of ¾ (~25 minutes)

Goals

- Participants will be able to interpret and explain multiple representations of a fraction.
- Participants will be able to use the concepts of the whole and equal parts to explain and critique explanations of representations of fractions.

Instructional sequence

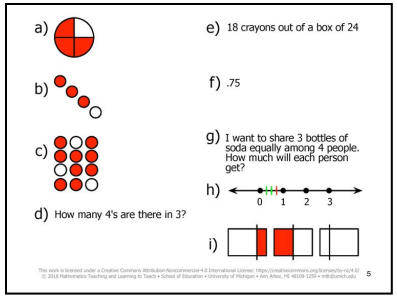
1. Introduce Part 4 and watch Video A.
2. Watch and discuss other teachers' responses (Videos B-F).

Resources

- Video A (02:13): Whole group discussion of item (a)
- Video B (00:32): Discussion of item (c)
- Video C (01:06): Discussion of item (d)
- Video D (01:50): Discussion of item (e)
- Video E (01:21): Discussion of item (g)

Supplements

- Handout: Large versions of images
- Math notes: Analysis of representations of 3/4

Detailed description of activity	Comments & other resources
1. Introduce Part 4: Now that participants have analyzed the representations, they will watch video clips of discussions of particular representations. Have participants watch *Video A* in which there is a whole group discussion of representation (a).  Two foundational ideas about fractions come out of this discussion: identifying the whole and equal parts. As they watch the video, participants should consider the focus questions introduced in Part 3 as well as the following questions: • How is the explanation in the clip similar or different to your interpretation of the representation? • What ideas and issues arise across the discussions of the representations? • In what ways do these ideas relate to foundational ideas about fractions, in particular, to defining the whole and to equal parts?	*It may be useful to print large versions of the images (see Document: Large versions of images) to place on the board. Participants can use these images to share their thinking with the whole group.* *There will not be time to discuss ALL of the representations of ¾. At the end of the session, consider distributing the Math notes document, which contains explanations for each representation.* *Encourage participants to attend to the figures as mathematical objects, not through the lens of curricular objects. The representations in their grade levels are not always subtle, but sometimes are pretty prescribed. Encourage participants to look at the figures without worrying yet about if they apply to the curriculum at a specific grade level or whether they would be confusing for students.* *If participants mention that a particular representation would be difficult for students, you might acknowledge that considering student difficulty is an important part of teaching and that carefully analyzing representations is a central part of determining what might be difficult. You could then ask participants to explain how they think the representation does (or does not) represent three-fourths and have them explain what aspect of that interpretation might be mathematically complicated or tricky. Or, if they say that they would never use a representation in their classroom because they find it confusing, it can be helpful to ask what about the representation is confusing, and then point out that, in teaching, students and/or textbooks often use representations that are unfamiliar and that one of the reasons that teaching is so mathematically demanding is that it requires being able to make sense of and think flexibly about representations different from ones you*

Detailed description of activity	Comments & other resources
Have participants discuss briefly with a partner or in whole group.	*might use if you were doing a problem yourself.* *Participants might also say that they don't like something about a representation—for example, they might argue that .75 should be written as 0.75. This type of comment can be used as an opportunity to point out that noticing and analyzing such nuanced aspects representations (e.g., aspects that don't make a difference mathematically, but could matter for teaching), is exactly the type of careful attention to features of representations that teachers have to engage in all the time.* *You might point out that what is meant by "equal parts" depends on the type of representation. For example, in an area model such as (a), equal parts means that each of the parts has the same area; however in a set model such as (b) or (c) equal parts means that each of the parts has the same number of objects.* *CCSSM Link: Throughout the elementary grades, CCSSM call for students to be engaged in critiquing the reasoning of others (Mathematical Practice #3).*
2. Select *additional videos* to view and discuss based on your context. By the end, make sure to tell participants that all representations are representations of ¾. If necessary share the Math notes document with explanations of representations that need further clarification. • Video B: Discussion of item (c) Possible discussion: Is this a complete explanation? If so, why? If not, what else needs to be done to clearly link it to ¾? • Video C: Discussion of item (d) Possible discussion: This item can be seen as a representation of ¾. What makes it tricky to interpret? • Video D: Discussion of item (e) Possible discussion: Can you use a variation of this explanation for representation (c)? • Video E: Discussion of item (g) Possible discussion: Where is one person's share in the diagram? Is this share ¾?	• *Video B: Discussion of item (c)* *Possible discussion: Is this a complete explanation? If so, why? If not, what else needs to be done to clearly link it to ¾?* *Possible response: No, this is not a complete explanation. The whole needs to be identified and the meaning of the 3 and the 4 in ¾ needs to be more clearly articulated.* • *Video C: Discussion of item (d)* *Possible discussion: This item can be seen as a representation of ¾. What makes it tricky to interpret?* *Possible response: Identifying the whole.* • *Video D: Discussion of item (e)* *Possible discussion: Can you use a variation of this explanation for representation (c)?* *Possible response: Yes. If you divide 24 items into four equal parts and shade three parts of those parts, then 18 circles are shaded.* • *Video E: Discussion of item (g)* *Possible discussion: Where is one person's share in the diagram? Is this share ¾?* *Possible response: Across the three long rectangles. Yes.*

Part 5: Explaining the representation: I want to share 3 bottles of soda equally among 4 people. How much will each person get? (~25 minutes)

Goals

- Participants will be able to understand diverse explanations for a fraction representation.
- Participants will be able to compare and critique explanations for a fraction representation.

Instructional sequence

1. Introduce Part 5 and watch Video A: Framing the problem.
2. Watch the sequence of video clips and discuss each video clip using focal questions.

Resources

- Video A (00:50): Framing the problem
- Video B (02:17): Representation 1
- Video C (01:13): Representation 2
- Video D (04:09): Representation 3
- Video E (01:53): Representation 4
- Video F (03:17): Comments on the representation

Supplements

- Video G (00:37): Practice interpreting a representation

Detailed description of activity	Comments & other resources
1. Introduce Part 5: In the professional development course, a debate arises about representation (g). The teachers have offered different explanations for how (g) can be seen as ¾. The teachers agree that (g) can represent ¾, but they disagree about whether this representation of ¾ corresponds to ¾ of one bottle or ¾ of three bottles. Have participants watch *Video A* in which Dr. Ball summarizes this debate. Allow participants to independently think about what they think the whole is.	
2. Have participants watch the discussion unfold in the sequence of five *videos* (B-F). Ask participants to consider how each teacher is using representations to try to communicate their ideas about (g). After each clip, participants should discuss with a partner: • How does the teacher's representation and explanation connect with (g) and/or other teachers' ideas? • What would you add or change about the explanation to make it more clear and/or complete? • Does the representation illustrate ¾ of *one* bottle or ¾ of *three* bottles?	*Participants may focus on whether or not (g) is a good representation of ¾. If this happens, remind participants that the task is to consider whether it is a possible representation of ¾, not whether or not it is a good representation.* • *Video B: Representation 1* *Each color indicates one person. Each person gets ¼ of each bottle. If you put the parts that are the same color together, you can see that each person gets ¾ of one bottle.* 

Detailed description of activity	Comments & other resources
	• *Video C: Representation 2* *After Katy's presentation, Natalie rearranges shaded parts.*  • *Video D: Representation 3* *This teacher uses a number line and adds drawing of a bottle under each whole number. He writes four students' names repeatedly in each whole, and he finally says each person has ¾ of one bottle.*  • *Video E: Representation 4* *This teacher uses three bottles as the whole. Each bottle contains 12 ounces. The teacher indicates that there are 36 ounces total and then divides this quantity by 4. The teacher concludes that one-quarter is 9 ounces and three-quarters are 27 ounces. However, he does not answer the question about what part each person gets. He says he made three-quarters of three bottles. He says he is not sure how to show the meaning of dividing into four.* • *Video F: Comments on the representation* *Amy comments on Dave's representation. The total is 36 ounces and 36 ounces is divided into four parts because four people are sharing. So, each person gets 9 ounces, and 9 ounces is three-quarters of one bottle (12 oz). It means also each person gets one-fourth of the whole three bottles. One-fourth of three is equivalent to three-fourths of one.*

Part 6: Types of connections with representations (~20 minutes)

Goals

- Participants will develop an awareness of different types of connections that can be made with representations.

Instructional sequence

1. Introduce Part 6 and watch Video A.
2. Watch the sequence of video clips (Videos B-E) and discuss each video clip.
3. (Optional) Watch some videos of teachers making connections and discuss.

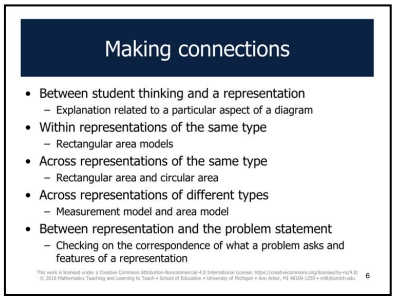
Resources

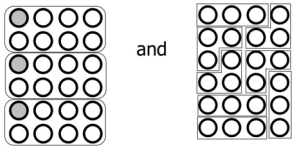
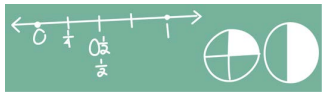
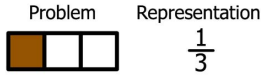
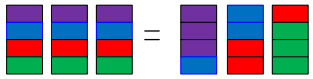
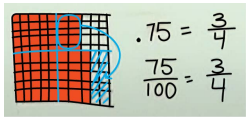
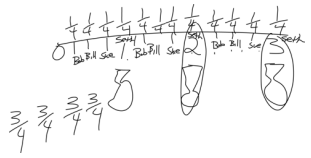
- Video A (02:22): Introducing types of connections
- Video B (00:46): Comparing 4/4 and 4/8
- Video C (00:37): Representing 1/8 of 24
- Video D (01:12): Representing ¼ and ½
- Video E (00:50): Representing the shaded part of the rectangle
- Transcript: Comparing four-fourths and four-eighths
- Transcript: One-eighth of 24
- Transcript: Representing one-fourth and one-half
- Transcript: Representing the shaded part of the rectangle

Supplements

- Handout: Connection in the session: Between participant thinking and a representation
- Video F (01:03): Connection in the session: Within representations of the same type
- Video G (02:06): Connection in the session: Across representations of different types
- Video H (04:09): Connection in the session: Between representations and the problem

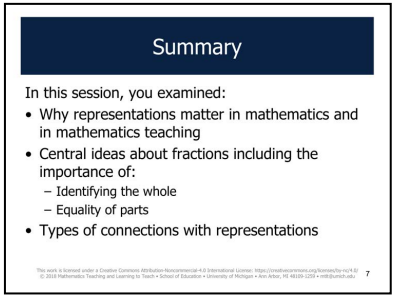
Detailed description of activity	Comments & other resources
1. Introduce Part 6: Using representations in teaching involves making different types of connections. Having a typology of connections can serve as a tool for learning about, supporting, and extending student understanding about when and where mathematical representations are used in instruction. Have participants watch *Video A* in which Dr. Ball introduces several types of connections that are made in teaching.	*The work in this module is designed to provide opportunities for participants to notice—and also to practice making—the types of connections introduced here. Developing these skills in the context of the module should enhance participants' abilities to make these types of connections with their students in their classrooms.*



Detailed description of activity	Comments & other resources
2. Have participants watch the sequence of videos (*B-E*) that show examples of children making each of the types of connections listed in the slide. Transcripts for all of these videos can be found in the Resources section. As participants watch each video, they should consider: • How does the clip illustrate the type of connection listed? • What examples of this type of connection have you seen in your work on the module? After watching each clip, discuss briefly in the whole group.	• *Video B: Comparing 4/4 and 4/8* *This connection is between student thinking and a representation.*  • *Video C: Representing 1/8 of 24* *This connection is within representations of the same type.*  • *Video D: Representing ¼ and ½* *This connection is across representations of different types.*  *Video E: Representing the shaded part of the rectangle* *This connection is between a representation and the problem.* 
3. (Optional) Participants might notice that they are making these same types of connections as they engage in this module. There are several examples of teachers' making these connections in their learning in the Supplements section. • Handout: Connection in the session: Between participant thinking and a representation • Video E: Within representations of the same type • Video G: Across representations of different types • Video H: Between representations and the problem	• *Video F: Within representations of the same type*  *After the prior presentation, this teacher combines same color parts and rearranges shaded parts. This participant is connecting her area model with another area model that has already been made.* • *Video G: Across representations of different types* *This teacher uses an area model, decimals, and fractions.*  • *Video H: Between representations and the problem* *This teacher uses number line, a discrete model, and numbers.* 

Part 7: Wrap up (~5 minutes)

Goals	***Instructional sequence***	***Resources***
• Participants will understand ways of connecting the session content to their classroom.	1. Summarize the work of the session. 2. Explain and distribute the Classroom Connection Activities.	

Detailed description of activity	Comments & other resources
1. Summarize the work of the session for participants. In this session, participants examined: • Why representations matter in mathematics and in mathematics teaching • Central ideas about fractions including the importance of identifying the whole and equality of parts • Types of connections between representations including: between student thinking and a representation, within representations of the same type, across representations of the same type, across representations of different types, and between a representation and the problem statement. 	
2. Distribute the *handout* you customized with selected Classroom Connection Activities and accompanying documents described below. <u>Required:</u> • Create and analyze representations of 4/3 to practice and extend the in-session work. • Make notes about how representations are connected in the classroom to connect the session work to classroom practice. • Analyze curriculum materials to identify representations used to support the learning of fractions and comparing fractions. <u>Optional:</u> • Read the Watanabe (2002) article, "Representations in Teaching and Learning Fractions" on tools and methods for representing fractions, fraction notation, and language.	*Stress to participants that CCA's related to teaching practices can be done with whatever mathematics is currently being taught. In other words, it is not necessary to be teaching fractions to be engaging in thinking about how representations are connected in the classroom.* *It may be helpful to ask participants to upload their responses or materials related to the Classroom Connection Activities so that you can review what participants have been thinking and trying prior to the next sessions. In addition, uploading responses would allow participants to easily share their ideas with each other.*