Description of the Session 3: Connecting representations and developing a working definition of a fraction

Session 3 extends the work on representing fractions that started in Session 2. In this session, participants first have a conversation from a CCA from the last session focused on the different ways in which fractions content appears in curriculum materials across grade levels. Participants examine connections between representations of ¾ in order to (a) deepen their own understanding of the mathematics, and (b) see the utility of connecting representations as a tool for supporting students' learning. Later, participants develop a working definition of a fraction in order to (a) deepen their own understanding of fractions, (b) to understand a definition for fractions that could be used with elementary school students, and (c) see the utility of generating (and making public) definitions of mathematics ideas.

Activities and goals of the session

Activities	Times	Corresponding parts of the session	Goals
Conversation about a CCA from last session	10 minutes		• Participants will know the different ways in which fraction content appears in curriculum across grade levels. • Participants will learn different ways of completing CCAs.
I. Preview	5 minutes	Part 1	• Participants will be oriented to the work of the session.
II. Connecting representations of fractions	60 minutes	Parts 2, 3 & 4	• Participants will be able to: o use the concepts of the whole and equal parts to explain representations of fractions; o compare and make connections between different representations of the same mathematical idea; o deepen their understanding of a particular representation by making connections to other representations; and o articulate reasons why making connections across multiple representations is important in elementary teaching.
III. Developing a working definition of a fraction	40 minutes	Parts 5 & 6	• Participants will begin to understand the role of definition in developing shared understanding and in supporting reasoning in mathematics. • Participants will be able to identify core elements of the concept of a fraction. • Participants will understand and be able to use a definition of a fraction to explain different representations of fractions.
IV. Wrap up	5 minutes	Part 7	• Participants will understand ways of connecting the session content to their classroom.

Classroom Connection Activities

Recommended	Optional
Type of task: Practice and extension of in-class work Description: Use of working definition of fractions to explain representations of ¾ and explain articulate the connection between identified pairs of representations of ¾ Type of task: Describing practice Description: Make notes about important features of numbers lines and how number lines are used in teaching	Type of Task: Reading – Bass & Ball (2009) Description: paper on definitions and defining in mathematics

Preparing for the session

☐ Make copies as needed:

- *Resources:* Handout: Representations of ¾ (Part 2); Handout: A working definition of a fraction (Part 6);

- *Supplemental resources:* Handout: Large versions of images (Part 2); Math notes: Analysis of multiple representations of ¾ (Part 2)

☐ Customize the Classroom Connection Activities and make copies as needed

☐ Test technical setups (Internet connection, speaker, projector)

Developing a culture for professional work on mathematics teaching (ongoing work of the facilitator throughout the module)

1. Encourage participation: talking in whole-group discussions; rehearsing teaching practices; coming up to the board as appropriate.

2. Develop habits of speaking and listening: speaking so that others can hear; responding to others' ideas, statements, questions, and teaching practices.

3. Develop norms for talking about teaching practice: close and detailed talk about the practice of teaching; supporting claims with specific examples and evidence; curiosity and interest in other people's thinking; serious engagement with problems of mathematics learning and teaching.

4. Develop norms for mathematical work:

   a) Reasoning: explaining in detail; probing reasons, ideas, and justifications; expectation that justification is part of the work; attending to others' ideas with interest and respect.

   b) Representing: building correspondences and making sense of representations, as well as the ways others construct and explain them.

   c) Carefully using mathematical language.

5. Help participants make connections among module content and develop the sense that this module will be useful in helping them improve their mathematics teaching, their knowledge of mathematics, their understanding of student thinking, and their ability to learning from their own teaching.

6. Help participants understand connections between module content and the Common Core Standards for School Mathematics.

Scope of the module *(focal content of this session in **bold**)*

Mathematics	Student thinking	Teaching practice	Learning from practice
• **representing fractions** • **defining fractions** • using and explaining methods and representations for comparing fractions • understanding how equivalence (of fractions) can be represented and used	• identifying and analyzing student conceptions, explanations, and representations of fractions • identifying and analyzing student strategies for comparing fractions	• selecting and generating representations • **connecting representations** • narrating the process of representing • supporting students in narrating the use of a representation • recording contributions and emerging mathematical ideas	• studying public recording space to learn from practice • using a conceptual framework to guide the planning, use, and analysis of public recording space

Conversation about a Classroom Connection Activity from last session (~10 minutes)

Goals	*Instructional sequence*	*Resources*
• Participants will know the different ways in which fraction content appears in curriculum across grade levels. • Participants will learn different ways of completing CCAs.	1. Participants discuss the examples of fraction content found in their curriculum materials. 2. Participants share different methods and materials that were useful when considering the CCAs.	Curriculum examples that participants bring

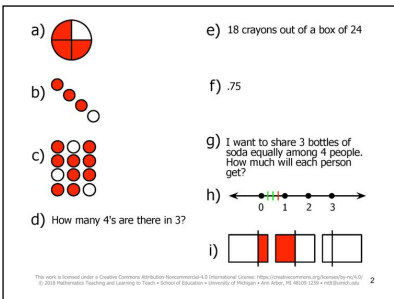| Detailed description of activity	Comments & other resources

1. In this session, participants will focus on the CCA that asked them to examine their curriculum materials. Ask participants to share examples they found focusing on:
 	
• **Essential ideas about fractions that are represented** (attention to the whole, equal partitions, congruence of fractional parts, equivalence):
 	
   o Are the wholes the same?
   o Are the parts equal? Is any attention given to representations in which the parts are not equal as a means to emphasize that equal parts are required?
   o Are all equal parts congruent? If the equal parts are not congruent, how are the parts shown to be equivalent?
 	
• **Models of fractions that appear** (area, set, linear):
 	
   o What models are used to represent fractions? How do the representations differ by grade level?
   o Do students have an opportunity to create representations or do they just use someone else's (the teacher's or the curriculum materials') representations?
 	
• **Comparison of fractions** (the use of models to illustrate comparisons, the comparison strategies shown):
 	
   o How are comparisons made (concretely or by sight)?
   o When pictorial representations are used, what method is used to make the comparisons evident?
 	
If participants are using different curriculum materials or if they are teaching at different grade levels, support participants in comparing and contrasting the representations that are used in the different curriculum materials they bring.
	Attend to the way fractions appear across grade levels and the progression of ideas that is illustrated in the examples that participants bring. Though there should be ample use of representations of fractions in the earlier grades, a major point for upper grade teachers is when and how representations are used at all. It is important for teachers to identify when their curriculum materials start to depend on abstract ideas and lessen their use of concrete or pictorial representations.
	
	Note the particular numbers that are illustrated in the materials. Do they tend to be low numbers, even numbers, numbers that involve 1, etc.?
	
	In general, it is rare to see representations in curriculum materials where the parts are not equal in size.

2. Take time to talk with participants about their experience of engaging with the CCA tasks. Encourage participants to share the different ways they went about completing the CCAs (with a grade level colleague, using the CD version of the teacher's manual, during a particular time of the school day, etc.).	*The goal of this conversation is to provide images of different ways in which CCAs can be done meaningfully and manageably between sessions.* *If you hear about difficulties in fitting the completion of CCAs into the daily work of teaching, use contributions from participants and your own insights to help participants generate options that can support CCA work.*

Part 1: Preview (~5 minutes)

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
• Participants will be oriented to the work of the session.	1. Introduce the session and watch the introductory video.	• Video (02:18): Introduction to session

Detailed description of activity	Comments & other resources
1. Introduce the session: Session 3 builds on and extends the work on representing fractions that was started in Session 2. In this session, participants further examine connecting representations, which sets the stage for considering a fundamental mathematical idea: the definition of a fraction. Have participants watch the *video* in which Dr. Ball presents an overview of the content that will be covered in Session 3.	

Overview of Session 3

• Connecting representations of fractions
• Developing a working definition of a fraction

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Part 2: Connecting representations of ¾ (~20 minutes)

Goals

- Participants will be able to compare and make connections between different representations of the same mathematical idea.
- Participants will be able to use the concepts of the whole and equal parts to explain representations of fractions.

Instructional sequence

1. Introduce Part 2 and watch the video in which Dr. Ball launches the task: select two representations and connect them.
2. Have participants work in pairs to articulate the connections between representations.

Resources

- Video (00:22): Introduction to task
- Handout: Representations of ¾

Supplemental:

- Handout: Large versions of images
- Math notes: Analysis of multiple representations of ¾

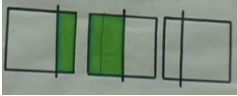| Detailed description of activity	Comments & other resources
1. Introduce Part 2: Connecting mathematical ideas to a particular representation or across representations is a foundational aspect of teaching mathematics. In the previous session, participants considered multiple representations of ¾ and began to consider the types of connections that can be made between representations in order to support sense-making. In this session, participants will practice making connections across different representations of the same mathematical idea. All of the representations <u>can</u> be interpreted as ¾. As discussed in the last session, when explaining how a representation corresponds to ¾, it is important to identify the whole and explain how the whole is being partitioned into equal parts. Have participants watch the *video* in which Dr. Ball introduces the task – making connections between two of the representations of ¾.	*There are many reasons why teaching skill in connecting representations supports student learning about mathematics. You will engage participants in thinking about these reasons toward the end of this session.* *You might motivate the work on making connections by pointing out that one of the NCTM Process Standards is "connections."* *CCSSM Link: In working on this activity, participants are engaged in several of the CCSSM mathematical practices including: making sense of problems and persevering in solving them (#1), reasoning abstractly and quantitatively (#2), constructing viable arguments and critiquing the reasoning of others (#3), and precisely communicating their thinking to others (#6).*

Detailed description of activity	Comments & other resources
2. Explain the directions on the slide. You may want to distribute the *Math notes: Analysis of representations of ¾* document that explain the different interpretations. Have participants work with a partner to articulate the connections between representations of ¾. Participants should note which representations they chose to connect and why. Encourage participants to consider how they used foundational ideas about fractions, such as identifying the whole and equal partitioning, to support the connections.	*Support participants in making the link between the language used in the Slide: Making connections (session 2) (see supplements) and the work that participants are doing in this part. For example:* • *Items (a) and (b): making connecting across representations of different types* • *Items (b) and (c): making connections within representations of the same type* *Possible pairs of fractions and reasons for selection which participants might consider are:* • *Items (a) and (b): The whole in (a) (which is one circle) and the whole in (b) (which is a set of four circles) are each divided into four equal parts and three of those parts are shaded.* • *Items (a) and (h): The whole (one circle/the interval from 0 to 1) is divided into four equal parts and three are marked.* • *Items (a) and (f): The shaded area of (a) is 75% or 0.75.* • *Items (b), (c), and (e): These are all set models of fractions.* • *Items (b) and (f): Three quarters are 75 cents.* • *Items (d) and (g): These are division models of fractions.* • *Items (d) and (i): There are three rectangles divided into four equal parts (amounts).* • *Items (a) and (i): These are both area models of fractions.* *Participants might have a tendency to explain why each representation is ¾ rather than focusing on connecting two representations of ¾. Encourage participants to make connections between two representations and identify how they are related to each other.*

The slide in the left column reads:

Connecting representations task

- Make connections between two of the representations from the ¾ activity
- Articulate connection between the pair of representations as if talking to a student and connect the pair back to the idea of ¾

Part 3: Explaining representation (i) (~25 minutes)

Goals	Instructional sequence	Resources
• Participants will deepen their understanding of a particular representation by making connections to other representations. • Participants will be able to compare and make connections between different representations of the same mathematical idea. • Participants will be able to use the concepts of the whole and equal parts to explain representations of fractions.	1. Introduce Part 3. 2. Record connections between items (g) and (i) and then watch Video A. 3. Watch Videos B & C in which teachers interpret item (i) and discuss the focal question for each video and then watch Video E, which connects different interpretations of (i) as ¾.	• Video A (01:23): Four people sharing three bottles • Video B (01:54): Green pieces vs. white pieces • Video C (03:09): What is the whole? • Video D (04:24): Connecting different interpretations

Detailed description of activity	Comments & other resources
1. Introduce Part 3: This part continues the work on making connections across representations by examining different explanations for representation (i). Making connections across different representations can support sense-making by enhancing understanding of each of the individual representations being connected. Explain to participants that they will be watching videos in which the attempt to make connections between representations (i) and (g) enabled teachers to develop a better understanding of representation (i).	
2. Have participants make notes (or discuss in small groups) the connections they see between representations (g) and (i). Then have participants watch *Video A* and note the similarities and differences they see between what the teacher in the video does and what they themselves said and did. Have participants discuss the focal question: In this interpretation, which is ¾ - the shaded or unshaded?	*Video A:* *Kathy interpreted the problem by comparing it with (e) and showing that each person got ¾ of one bottle. In this interpretation, the green pieces represent ¾.*

Detailed description of activity	Comments & other resources
3. Have participants try to unpack the mathematics of representation (i) in greater depth. Watch *video B or C* and then *video D,* in which the teachers discuss representation (i). After watching each clip, participants should: • Makes notes about the given focal question. • Record the ways in which the speaker identifies the whole, uses equal parts, and/or draws upon other foundational ideas about fractions. For each video, allow time to discuss (in small or whole group) responses to the focal questions and the use of foundational ideas about fractions before moving to the next video. *Video B: Green pieces vs. white pieces* *Focal question*: What is the whole in this interpretation? *Video C: What is the whole?* *Focal question*: What is the whole in this interpretation? *Additional question:* How many equal parts make up the whole? How many of those parts are green? *Video D: Connecting different interpretations* *Focal question*: Explain ¼ of 3, ¾ of 1, and ¾ of 3 in this representation of ¾.	 *Video B:* *Katy divided each rectangle into four equal parts and showed that the first green part is 1/4 and the second green part is 2/4. She added 1/4 + 2/4 to show that 3/4 are shaded. In this interpretation, the whole is one rectangle.* *Video C:* *Janet cut the larger green part into half and found three equal parts, but was unclear on what the whole would be (one whole is three rectangles or each rectangle is one whole). Teri interpreted one rectangle as the whole and connected the representation to ¾ of 1 bottle in item (g). Teri later says that the green can be interpreted as twelfths, which implies that the whole is comprised of all three rectangles.* *Video D:* *Teachers discuss different interpretations of (i).* • *Each person in item (i) has ¾ of 1 box.* • *The white pieces are ¾ of 1 box and the green pieces are ¼ of 3 boxes.* • *The green pieces are ¾ of 1 rectangle (or ¼ of 3 rectangles) and the white pieces are ¾ of 3 rectangles.* • *¼+¾=1. If ¼ is shaded, ¾ is not shaded. ¼ of 3 rectangles is shaded and ¾ of 3 is not shaded.*

Part 4: The benefits of connecting representations (~15 minutes)

Goals	Instructional sequence	Resources
• Participants will be able to articulate reasons why making connections across multiple representations is important in elementary teaching.	1. Introduce Part 4 and discuss the benefits of making connections across representations when teaching. 2. Watch the video and consider other teachers' reactions. 3. Discuss ways to use multiple representations in teaching and why this is important.	• Video (02:18): Teachers' considerations

Detailed description of activity	Comments & other resources

1. Introduce Part 4: In this part of the session, participants will discuss the benefits of making connections between representations and then relate this to their own classroom practice.
 	
Before watching the video, have participants record (or discuss) responses to the following question: *Why is making connections across multiple representations important in teaching mathematics?*
 	
Making connections with representations
• Between student(s) thinking and a representation
  – Explanation related to a particular aspect of a diagram
• Within representations of the same type
  – Rectangular area models
• Across representations of the same type
  – Rectangular area and circular area
• Across representations of different types
  – Measurement model and area model
• Between representation and the problem statement
  – Checking on the correspondence of what a problem asks and features of a representation
• Connecting mathematical language and ideas to representations
  – Using subject matter terminology and ideas to name and describe aspects of representations
	Teachers need to be able to make connections, as well as support students in making them. The two skills are related, but not the same. This module supports teachers in doing both kinds of work with connections.
	
	If time in this session is tight, consider skipping Part 4 if you discussed these ideas in Session 2.
	
	Participants may find it helpful to refer back to two slides from Session 2 (Slide: Representations matter in mathematics & Slide: Using representations in teaching mathematics) to support their considerations.

2. Have participants watch the *video* in which Dr. Ball and teachers discuss the benefits of connection representations in terms of: • *Mathematics:* deepening understanding; facilitating mathematical thinking; highlighting key mathematical ideas • *Teaching practice:* explaining mathematical concepts; broadening participation • *Student thinking:* encouraging multiple solution methods; providing access to diverse learners; building on prior knowledge	*Other examples of reasons why teachers need to be skilled in making connections include:* • *When several representations have been shared, making explicit connections between them helps students to make sense of the ongoing dialog.* • *Teachers need to help students link their own ways of representing to others that might be of use.* • *Teachers need to help students see that making connections is a routine part of what one does when engaging in mathematics.*

Detailed description of activity	Comments & other resources
3. Discuss in whole group the ways in which participants have used or would like to use multiple representations to support student learning in their own classroom.	*Possible questions to pose:* • *How do your curriculum materials introduce (or use) multiple representations of fractions?* • *How do your curriculum materials make connections between representations?* • *What representation of fractions is most familiar (to you or in your curriculum materials)? What representation is least familiar (to you or in your curriculum materials)?* *You might link this discussion to the Classroom Activities work from Session 2, which asked participants to note the types of connections they made in their classrooms.*

Part 5: Developing a definition of a fraction (~20 minutes)

Goals	**Instructional sequence**	**Resources**
• Participants will begin to understand the role of definition in developing shared understanding and supporting reasoning in mathematics. • Participants will be able to identify core elements of the concept of a fraction.	1. Introduce Part 5 and discuss the meaning of fraction. 2. Watch the video in which Dr. Ball and teachers develop a definition of a fraction. 3. Discuss how the initial definition has developed.	• Video (11:49): Developing a definition of a fraction

Detailed description of activity	Comments & other resources
1. Introduce Part 5: Making connections between representations of fractions depends on a shared understanding of what a fraction is. Establishing and using definitions is a way to develop shared meaning of mathematical ideas. In this part of the session, participants will consider and discuss a working definition of a fraction. Ask participants to record their responses to the following questions: • How would you define a fraction? • How would you explain what a fraction is to an elementary student? Have participants share their notes with a colleague.	*"Working" is used to signal that the definition could be further developed and revised as it is used.* *You might comment about the importance of definitions in both mathematics and mathematics teaching.* • *In mathematics: Definitions are one of the foundations of mathematical reasoning. Mathematical definitions provide the precision and shared meaning that is crucial for effective mathematical communication and reasoning.* • *In mathematics teaching: Knowledge of definitions supports the inspection of instructional materials and making judgments about the quality and usability of those materials. Teachers can use their knowledge of definitions to support communication, mathematical reasoning, and the reconciliation of classroom disagreement about mathematics.* *It is important that participants record their responses to these questions as they will compare their responses to a working definition that was developed by Dr. Ball and the teachers attending the professional development course.*

Detailed description of activity	Comments & other resources

2. Have participants watch the video in which the teachers in the professional development course develop a working definition of a fraction.
 	
As participants watch the video, have them consider the following focus questions:
 	
• What aspects of the working definition developed in the video correspond with the ideas you recorded or discussed with your colleague? How are they similar or different?
 	
• What aspects of the working definition seem difficult to develop? What do you think makes these ideas hard to articulate or establish?
	Summary of video:
	
	At the beginning of the video, Dr. Ball explains the importance of definitions. Students often use the same term but attach different definitions to that term. For example, "even number" might be defined by one student as "a number which can be split into 2 equal parts" and by another student as "every other number on the number line."
	
	As the video progresses, the teachers then brainstorm the following as elements of the definition of a fraction:
	
	• *Figuring out what the whole is*
	• *Divided into equal parts / sharing equally*
	• *Numerator & denominator*
	   ○ *Numerator: a top portion which represents parts which are less than the whole*
	   ○ *Denominator: representing the whole*
	   ○ *Refining the definition of numerator and denominator using 8/5*
	   ○ *Numerator: How many equal parts of the whole*
	   ○ *Denominator: Part of the whole, the number of equal pieces (parts) in the whole / The number of equal parts that make up the whole*
	
	The video is rather long, so you might consider pausing it at strategic points to ask participants to reflect on the ideas being raised. For example, you could stop at 2:03 to check in with participants about why definitions are important or to ask about participants' own work with definitions and what functions they serve. You might also stop at 4:50 and 8:07 and ask participants to try to answer the question Dr. Ball poses.

3. Discuss the focus questions in whole group: • What aspects of the working definition developed in the video correspond with the ideas you recorded or discussed with your colleague? How are they similar or different? • What aspects of the working definition seem difficult to develop? What do you think makes these ideas hard to articulate or establish?	

Part 6: A working definition of a fraction (~20 minutes)

Goals	Instructional sequence	Resources
• Participants will understand and be able to use a definition of a fraction to explain different representations of fractions.	1. Introduce Part 6 and watch the video that presents a working definition of a fraction. 2. Use the definition to explain representations for fractions.	• Video (04:28): Working definition of a fraction • Handout: A working definition of a fraction

Detailed description of activity	Comments & other resources
1. Introduce Part 6: This part further develops a working definition of a fraction and then provides practice applying it to specific examples. Set up the video: The teachers in the professional development continued to develop their working definition of fractions. During their discussion, additional ideas were added to the definition, including notation for the numerator (n) and the denominator (d), and the condition that the denominator cannot equal 0. Have participants watch the *video* in which Dr. Ball consolidates their discussion by presenting a working definition that will be used as the basis for further work in this module.	Consider pausing the video at the following points to ask participants to reflect on the ideas raised in the video. • *01:32 – Ask participants to apply identifying the whole, make d equal parts, and 1/d shows one of the parts to the fraction 1/5.* • *02:46 – Ask participants to apply, "If you have n of 1/d then you have n/d" to the fraction 8/5 and then to a representation of 8/5 like an area model or a number line.* <u>CCSSM Link</u>: The Definition of a Fraction used in this module closely aligns with the definition of a fractions used in the CCSSM. The CCSSM definition is: "understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b." This definition is introduced in Grade 3, as is an application to the number line (3.NF.1 and 3.NF.2).
2. After watching the clip, have participants consider and discuss the following with a partner: • Try to apply the working definition in the slide to the examples 2/5 and 8/5 that came up earlier in the discussion. As time permits, elicit comments in whole group. *If time permits, consider discussing the following questions:* • Can the working definition be used for different representations of fractions (e.g., area model, sets of objects, number line)? • How would you revise the working definition of fraction? For instance, how might you attach the language of numerator or denominator to the definition?	*For example: To represent the fraction 2/5, we first identify our whole. In this case, our whole is this rectangle.* *Then, make 5 equal parts. We can write 1/5 to show one of those parts. If you have 5 of 1/5, then you have the whole.* *If you have 2 of those equal parts shaded, then 2/5 of the rectangle is shaded.*

Slide content:

A working definition of a fraction
− Identify the whole
− Make d equal parts
− Write $^1/_d$ to show one of the equal parts
− If you have d of $^1/_d$, then you have the whole
− If you have n of $^1/_d$, then you have $^n/_d$
• n and d are whole numbers
• $d \neq 0$

Part 7: Wrap up (~5 minutes)

Goals	Instructional sequence	Resources
• Participants will understand ways of connecting the session content to their classroom.	1. Summarize the session. 2. Explain the Classroom Connection Activities.	

Detailed description of activity	Comments & other resources

1. Summarize the session: This session focused on articulating key ideas about fractions. These ideas were used to make connections across representations and to develop a working definition of a fraction. The session also identified benefits of using multiple representations in elementary teaching.
 	
Summary
 	
In this session, you continued to work on:
• The mathematics used in elementary teaching
  – Articulating key ideas about fractions
  – Formulating and analyzing definitions
• Teaching practices used to help elementary students learn mathematics
  – Making connections across representations in order to further understanding
• Considering the benefits of using multiple representations in the classroom	

2. Distribute the handout you customized with selected *Classroom Connection Activities* and accompanying documents described below. Required: • Use the *working definition of a fraction* to explain several representations of ¾ and explain the connections between several representations. • Make notes about important features of numbers lines and how number lines are used in teaching. Optional: • Read the Bass & Ball (2009) article, "Definitions and Defining in Mathematics and Mathematics Teaching."	*Classroom Connection Activities support extended engagement in professional development content, as well as ways to connect and try out some of the ideas and practices that were a part of the session in their own classrooms.* *It may be helpful to ask participants to upload their responses or materials related to the Classroom Connection Activities so that you can review what participants have been thinking and trying prior to the next sessions. In addition, uploading responses would allow participants to easily share their ideas with each other.*