

***Description of the Session 4: Using the number line***

In this session, participants will first have a conversation about a CCA from the last session focused on connecting representations of  $\frac{3}{4}$ . Session 4 extends the work articulating key ideas about fractions and developing a working definition of a fraction that started in Session 3 through consideration of the number line. Participants use a video record as a text for developing skill in analyzing student thinking, in this case about using a number line to compare two fractions. Participants also work to establish an understanding of the key properties and conventions of the number line. Finally, participants apply their ideas on the definition of a fraction, number lines, and interpreting student thinking about representations to anticipate, analyze, and develop strategies for responding to student thinking about labeling locations (points) on a number line.

***Activities and goals of the session***

<b>Activities</b>	<b>Times</b>	<b>Corresponding parts of the session</b>	<b>Goals</b>
Conversation about a CCA from the last session	10 minutes		<ul style="list-style-type: none"> <li>Participants will develop skill in using key ideas about fractions to make mathematical connections.</li> </ul>
I. Preview	5 minutes	Part 1	<ul style="list-style-type: none"> <li>Participants will be oriented to the work of the session.</li> </ul>
II. Analyzing student thinking	20 minutes	Part 2	<ul style="list-style-type: none"> <li>Participants will begin to develop skill in attending to student thinking about the use of number lines to solve a fraction comparison task.</li> </ul>
III. Exploring properties and conventions of the number line	55 minutes	Parts 3, 4 & 5	<ul style="list-style-type: none"> <li>Participants will be able to describe the location of numbers on a number line and analyze others' strategies.</li> <li>Participants will be able to identify and use key properties and conventions of the number line, as well as additional observations about the line.</li> </ul>
IV. Analyzing students' errors when labeling marked points on the number line	25 minutes	Part 6	<ul style="list-style-type: none"> <li>Participants will be able to apply their ideas on the definition of fraction, number lines, and interpreting student thinking about representations to anticipate, analyze, and develop strategies for responding to student thinking about labeling locations (points) on a number line.</li> <li>Participants will be able to identify 5 common errors students make when labeling locations (points) on a number line.</li> </ul>
V. Wrap up	5 minutes	Part 7	<ul style="list-style-type: none"> <li>Participants will understand ways of connecting the session content to their classroom.</li> </ul>

***Classroom Connection Activities***

Required	Optional
<p>Type of task: Analysis of curriculum materials Description: Identify treatment and features of number lines used in curriculum.</p> <p>Type of task: Practice and extension work Description: Analyze student work samples and anticipate the reasoning that student may have used to locate numbers on the number line.</p>	<p>Type of task: Reading Description: Excerpt on geometric representations of the number line from National Research Council's (2001) report, <i>Adding it up: Helping children learn mathematics</i>.</p>

***Preparing for the session***

- Colored pencils, rulers
- Make copies as needed:
  - *Resources:* Transcript: Betsy's use of the number line (Part 2); Handout: Focus questions (Part 2); Handout: Blank number line (Part 3); Handout: Summarizing key ideas about the number line (Part 5); Handout: Number line task (Part 6); Handout: Working definition of a fraction (Slide 6)
  - *Supplemental resources:* Math notes: Properties and conventions of the number line (Part 5); Math notes: The strategies underlying students' errors when identifying points on the number line (Part 6)
- Customize the Classroom Connection Activities and make copies as needed
- Test technical setups (Internet connection, speakers, projector)

***Developing a culture for professional work on mathematics teaching (ongoing work of the facilitator throughout the module)***

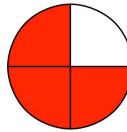

1. Encourage participation: talking in whole-group discussions; rehearsing teaching practices; coming up to the board as appropriate.
2. Develop habits of speaking and listening: speaking so that others can hear; responding to others' ideas, statements, questions, and teaching practices.
3. Develop norms for talking about teaching practice: close and detailed talk about the practice of teaching; supporting claims with specific examples and evidence; curiosity and interest in other people's thinking; serious engagement with problems of mathematics learning and teaching.
4. Develop norms for mathematical work:
  - a) Reasoning: explaining in detail; probing reasons, ideas, and justifications; expectation that justification is part of the work; attending to others' ideas with interest and respect.
  - b) Representing: building correspondences and making sense of representations, as well as the ways others construct and explain them.
  - c) Carefully using mathematical language.
5. Help participants make connections among module content and develop the sense that this module will be useful in helping them improve their mathematics teaching, their knowledge of mathematics, their understanding of student thinking, and their ability to learning from their own teaching.
6. Help participants understand connections between module content and the Common Core Standards for School Mathematics.

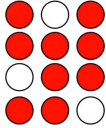
***Scope of the module (focal content of this session in bold)***

<b>Mathematics</b>	<b>Student thinking</b>	<b>Teaching practice</b>	<b>Learning from practice</b>
<ul style="list-style-type: none"> <li>• <b>representing fractions</b></li> <li>• defining fractions</li> <li>• using and explaining methods and representations for comparing fractions</li> <li>• understanding how equivalence (of fractions) can be represented and used</li> </ul>	<ul style="list-style-type: none"> <li>• <b>identifying and analyzing student conceptions, explanations, and representations of fractions</b></li> <li>• identifying and analyzing student strategies for comparing fractions</li> </ul>	<ul style="list-style-type: none"> <li>• <b>selecting and generating representations</b></li> <li>• connecting representations</li> <li>• narrating the process of representing</li> <li>• supporting students in narrating the use of a representation</li> <li>• recording contributions and emerging mathematical ideas</li> </ul>	<ul style="list-style-type: none"> <li>• studying public recording space to learn from practice</li> <li>• using a conceptual framework to guide the planning, use, and analysis of public recording space</li> </ul>

**Conversation about a Classroom Connection Activity from last session (~10 minutes)**

<b>Goals</b>	<b>Instructional sequence</b>	<b>Resources</b>
<ul style="list-style-type: none"> <li>Participants will develop skill in using key ideas about fractions to make mathematical connections</li> </ul>	<ol style="list-style-type: none"> <li>Discuss participant responses to the CCA task focused on connecting particular representations of <math>\frac{3}{4}</math>.</li> </ol>	<ul style="list-style-type: none"> <li>Diagrams of representations (a), (c), (e), and (i) from the representations of <math>\frac{3}{4}</math> document</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Discuss the CCA task focused on making connections between representations of <math>\frac{3}{4}</math>. Representations (a) and (i) are both area models and representations (c) and (e) are both set models. There are many other ways of making connections between these diagrams. One way of making connections between the figures is to draw on key ideas in the working definition of a fraction and show how both figures illustrate its components.</p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center;"><b>A working definition of a fraction</b></p> <ul style="list-style-type: none"> <li>Identify the whole</li> <li>Make <math>d</math> equal parts</li> <li>Write <math>\frac{1}{d}</math> to show one of the equal parts</li> <li>If you have <math>d</math> of <math>\frac{1}{d}</math>, then you have the whole</li> <li>If you have <math>n</math> of <math>\frac{1}{d}</math>, then you have <math>\frac{n}{d}</math></li> </ul> <ul style="list-style-type: none"> <li><math>n</math> and <math>d</math> are whole numbers</li> <li><math>d \neq 0</math></li> </ul> </div> <p>Focus first on discussing figures (a) and (i), and then figures (c) and (e), listening to the connections that participants have made. Where relevant connect their points to the components in the working definition of a fraction. If some connections to the definition do not come up, you may want to raise them for consideration by asking some of the following questions:</p> <ul style="list-style-type: none"> <li>Where is the "whole" in each of the fractions?</li> <li>What does <math>\frac{1}{4}</math> look like in each?</li> <li>Where are three of those fourths in the representation?</li> </ul> <p>The work in this module is designed to provide opportunities to notice, and also practice making, the types of connections introduced here. Developing these skills in the context of the module should enhance participants' abilities to make these types of connections with their students in their classrooms.</p>	<p><i>Notes about connecting representations a and i:</i></p> <ul style="list-style-type: none"> <li><u>Identify the whole in each.</u> <ul style="list-style-type: none"> <li><i>Representation (a): The entire circle is the whole.</i></li> <li><i>Representation (i): Identifying the whole is more complicated. There are at least two possibilities and they lead to different "answers" as to what <math>\frac{3}{4}</math> is. The whole can be all three rectangles or the whole can be one rectangle. If the whole is one rectangle, then the red can be shown to be <math>\frac{3}{4}</math> through equal partitioning of that whole. If instead the whole is the set of three rectangles, then the white can be shown to be <math>\frac{3}{4}</math> through the partitions that are already made.</i></li> </ul> </li> <li><u>Identify where the four equal parts are and what one of them looks like.</u> <ul style="list-style-type: none"> <li><i>Representation (a): The whole is divided into four equal parts. One of these parts is <math>\frac{1}{4}</math>.</i></li> <li><i>Representation (i): If one rectangle is the whole, each rectangle can be partitioned into four equal parts. One of those parts will be the size of the red piece in the rectangle on the left of the diagram and the size of one of those parts is <math>\frac{1}{4}</math>. If the three rectangles is seen to be the whole, the representation is already partitioned into fourths, each the size of the white piece in the rectangle on the left side of the representation.</i></li> </ul> </li> <li><u>Identify where the three equal parts making <math>\frac{3}{4}</math> are.</u> <ul style="list-style-type: none"> <li><i>Representation (a): The 3 equal parts shaded red are the 3 in the <math>\frac{3}{4}</math>.</i></li> <li><i>Representation (i): If one rectangle is the whole, the three equal parts shaded red show the 3 in the <math>\frac{3}{4}</math>. If the three rectangles is the whole, the three white parts show the 3 in <math>\frac{3}{4}</math>.</i></li> </ul> </li> </ul> <div style="text-align: right; margin-top: 20px;">  <p>a)</p>  <p>i)</p> </div>

Detailed description of activity	Comments & other resources
	<p><i>Notes about connecting representations (c) and (e):</i></p> <ul style="list-style-type: none"> <li>• <u>Identify what is serving as the whole in each.</u> <ul style="list-style-type: none"> <li>○ Representation (c): Each column could be considered a whole or the entire representation could be considered the whole. There are four circles in each column.</li> <li>○ Representation (e): The entire box of 24 crayons represents the whole.</li> </ul> </li> <li>• <u>Identify where the four equal parts are and what one of them looks like.</u> <ul style="list-style-type: none"> <li>○ Representation (c): If the set of 12 circles represents the whole, then fourths can be made by separating the 12 circles into four equal groups that each contain three circles. Three of these groups should contain only red circles and one group should contain only white circles.</li> <li>○ Representation (e): The entire box of 24 crayons represents the whole. To make fourths, separate the 24 crayons into four equal groups that will result in 6 crayons in each group.</li> </ul> </li> <li>• <u>Identify where the three equal parts making <math>\frac{3}{4}</math> are.</u> <ul style="list-style-type: none"> <li>○ Representation (c): One group of 3 circles is <math>\frac{1}{4}</math>. This means that <math>\frac{3}{4}</math> is shown by 3 groups of 3 circles (or 9 circles). The 9 red circles represent <math>\frac{3}{4}</math>.</li> <li>○ Representation (e): Each fourth is made up of 6 crayons, meaning that <math>\frac{3}{4}</math> is shown by 18 crayons.</li> </ul> </li> </ul> <div style="text-align: right; margin-top: 20px;">  <p>○) 18 crayons out of a box of 24</p> </div>

**Part 1: Preview (~5 minutes)**

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
<ul style="list-style-type: none"> <li>Participants will be oriented to the work of the session.</li> </ul>	1. Introduce the session and watch the introductory video.	<ul style="list-style-type: none"> <li>Video (04:26): Overview of session</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce the session: Student thinking is central to mathematics teaching. Teachers need to learn what students think, nurture connections between what students think and established mathematical knowledge and practices, and support growth in thinking over time.</p> <div data-bbox="556 522 957 826" style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #003366; color: white; padding: 2px;"><b>Overview of Session 4</b></p> <ul style="list-style-type: none"> <li>Analyzing a student's use of the number line</li> <li>Exploring properties and conventions of the number line</li> <li>Analyzing students' errors when using the number line</li> </ul> <p style="font-size: 8px; margin-top: 5px;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach in School of Education at University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu</p> </div> <p>Have participants watch the <i>video</i> in which Dr. Ball presents an overview of the session.</p>	

**Part 2: Interpreting student thinking about the number line (~20 minutes)**

<b>Goals</b>	<b>Instructional sequence</b>	<b>Resources</b>
<ul style="list-style-type: none"> <li>Participants will develop skill in attending to student thinking about the use of number lines to solve a fraction comparison task.</li> </ul>	<ol style="list-style-type: none"> <li>Introduce Part 2 and provide context for the classroom video.</li> <li>Watch the classroom video (Betsy's use of the number line). Use the focus questions to guide viewing.</li> <li>Analyze the video – compile a list of ideas that Betsy appears to know about the number line using evidence from the video/transcript.</li> </ol>	<ul style="list-style-type: none"> <li>Video (02:30): Betsy's use of the number line</li> <li>Handout: Focus questions</li> <li>Transcript: Betsy's use of the number line</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce Part 2 and the use of video in the session: Student thinking is central to mathematics teaching. Teachers need to learn what students think, nurture connections between what students think and established mathematical knowledge and practices, and support growth in students' thinking over time. Video is one of many tools that provide a window into student thinking. In this part, participants analyze a video clip from a third-grade classroom to develop skill in attending to students' use of and thinking about the number line.</p> <div data-bbox="779 662 1163 950" style="border: 1px solid black; padding: 5px;"> <p style="text-align: center;"><b>Video clip from a third-grade lesson on fractions</b></p> <ul style="list-style-type: none"> <li>• Third graders (8 year-olds)</li> <li>• Diverse classroom, many English language learners</li> <li>• Early May (three weeks into a five-week fractions unit)               <ul style="list-style-type: none"> <li>– Emerged from study of division (problems with remainders)</li> <li>– Single objects as the whole (cookies, brownies, graham crackers)</li> <li>– Drawings (rectangles, circles)</li> <li>– Sets of objects as the whole (boxes of crayons)</li> <li>– Number line</li> <li>– Mapping one representation onto others</li> </ul> </li> </ul> <p style="font-size: small; text-align: center;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu 2</p> </div> <p>Have participants read the slide with the context for the classroom video. In this video, a third grader, Betsy, uses a number line to compare <math>\frac{4}{4}</math> and <math>\frac{4}{8}</math>.</p>	<p><i>Video can serve as a "text," displaying concrete instances of student thinking that can be slowed down, replayed, and shared. With video, student thinking can be seen, heard, and studied in more detail than can typically be done in real time. The use of video can also aid description and analysis: It allows people to support claims with evidence from things that were said and done rather than from impressions and recollections.</i></p>
<p>2. Introduce the focus questions that will guide the viewing of Betsy's use of the number line (see Slide: Focus questions).</p> <div data-bbox="779 1089 1163 1377" style="border: 1px solid black; padding: 5px;"> <p style="text-align: center;"><b>Focus questions</b></p> <ul style="list-style-type: none"> <li>• How does the student use the number line to solve the problem or explain her solution?</li> <li>• What does the student appear to understand about the number line?</li> </ul> <p style="font-size: small; text-align: center;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu 3</p> </div>	

Detailed description of activity	Comments & other resources
<p>3. Distribute the <i>transcript</i> of the classroom video. Have participants watch the <i>video</i> of Betsy's use of the number line.</p> <p>Encourage participants to use the transcript to follow along – or to write on it – if it is helpful. Participants should take notes that capture their thinking about the focus questions.</p>	<p><i>You may want to print the transcript in color.</i></p> <p><i><u>CCSSM Link:</u> The classroom video contains examples of third grade students engaged in two of the mathematical practices (#3 and #6). The student, Bernadette, is constructing a viable argument for why <math>\frac{4}{4}</math> is greater than <math>\frac{4}{8}</math> using a number line (#3). When communicating her reasoning to classmates, Bernadette carefully explained how she compared the two fractions using the number line ("<math>\frac{4}{8}</math> must be a lot less than <math>\frac{4}{4}</math> because <math>\frac{4}{8}</math> is – this close to zero and <math>\frac{4}{4}</math> is at one"). While not evidenced in the video, the segment concludes with other students being asked to critique Bernadette's reasoning (#3). Comparing two fractions with the same denominator is a Grade 3 standard (3.NF.3).</i></p>
<p>4. After watching the video, have participants work with a partner to compile a list of ideas that Betsy appears to know about the number line. Encourage participants to try to connect these ideas with evidence from the transcript or action in the video. Participants should note any concrete evidence of a potential misunderstanding.</p> <p>As time permits, have pairs share some of the ideas in whole group.</p>	<p><i>The work that participants are doing in this part provides a launching point for considering what ideas you might want to make available for students to learn about the number line.</i></p> <p><i>Some ideas that participants may note:</i></p> <ul style="list-style-type: none"> <li>• <i>Seems very proficient: able to divide into both fourths and eighths on the same number line</i></li> <li>• <i>Points for eighths and fourths didn't line up (e.g., <math>\frac{4}{8}</math> and <math>\frac{2}{4}</math>)</i></li> <li>• <i>Used closer to zero as part of her argument: She understood that because <math>\frac{4}{8}</math> is closer to zero (than <math>\frac{4}{4}</math>), it is the smaller number</i></li> </ul>



**Part 3: Locating numbers on the number line (~20 minutes)**

<b><u>Goals</u></b>	<b><u>Instructional sequence</u></b>	<b><u>Resources</u></b>
<ul style="list-style-type: none"> <li>Participants will be able to describe the location of numbers on a number line and analyze others' strategies.</li> </ul>	<ol style="list-style-type: none"> <li>Introduce Part 3 and discuss what is entailed in using representations in teaching.</li> <li>Complete the following activity for a set of fractions:               <ul style="list-style-type: none"> <li>Locate the number on a blank number line (and describe the location).</li> <li>Watch and reflect upon a video of a teacher describing the location of the number.</li> </ul> </li> </ol>	<ul style="list-style-type: none"> <li>Handout: Blank number lines</li> <li>Video A (00:49): Describing the location of <math>1\frac{1}{2}</math></li> <li>Video B (01:05): Describing the location of <math>\frac{1}{8}</math></li> <li>Video C (00:58): Describing the location of <math>\frac{1}{4}</math> and <math>\frac{3}{4}</math></li> <li>Video D (00:55): Describing the location of <math>-54</math> and other relatively large numbers</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce Part 3: Using representations in teaching involves being able to accurately construct representations, to see how different representations are applicable in particular situations, to use specialized language associated with the representation, as well as support students in constructing, interpreting, and communicating about representations. Doing this requires being able to talk about representations in ways that make the mathematics accessible for students to learn. In this part of the session, participants will look closely at what is involved in describing the location of numbers on the number line.</p>	<p><i>In the next two parts, participants will examine key properties and conventions of the number line.</i></p>
<p>2. Distribute number lines (handout) with -2, -1, 0 and 1 labeled. Participants should place <math>1\frac{1}{2}</math> on the line and explain how they figured it out to a partner.</p> <p>Have participants watch <i>Video A</i> in which a teacher describes how she would locate <math>1\frac{1}{2}</math> on the line. As participants watch the video, have them consider the following questions:</p> <ul style="list-style-type: none"> <li>What would you add to the explanation to make it more clear and/or complete?</li> <li>Is there anything you would change about how the work is recorded on the board?</li> </ul> <p>Briefly discuss questions in whole group.</p>	<p><i>This is a preview of work on narration that will be a focus of Session 5.</i></p> <p><i>For each video, be sure to have participants locate the number on their own number line and discuss its placement with a partner <u>before</u> watching the videos so that they can compare their ideas with the ideas of the teachers in the video.</i></p> <p><i>Video A (Describing the location of <math>1\frac{1}{2}</math>): A more complete explanation would state that <math>1\frac{1}{2}</math> is located halfway between 1 and 2 (or alternatively, the interval from 1 and 2 can be divided into 2 equal parts and <math>1\frac{1}{2}</math> is located at the end of the first part).</i></p>

Detailed description of activity	Comments & other resources
<p>3. Repeat the process in (2) for the following numbers: <math>\frac{1}{8}</math> (<i>Video B</i>); <math>\frac{1}{4}</math> and <math>\frac{3}{4}</math> (<i>Video C</i>); and <math>-\frac{5}{4}</math> (<i>Video D</i>).</p> <p>Discuss in pairs or in whole group as time permits.</p>	<p><i>Video B (Describing the location of <math>\frac{1}{8}</math>): There is a mismatch between what is said ("divide between the 0 and 1 into 4 pieces") and what is marked (interval from 0 and 1 is divided into 8 equal parts).</i></p> <p><i>Video C (Describing the location of <math>\frac{1}{4}</math> and <math>\frac{3}{4}</math>): The explanations do not connect with interpreting the numerator and denominator. To mark <math>\frac{1}{4}</math>, we need to divide the interval from 0 and 1 into 4 equal parts. The fraction <math>\frac{1}{4}</math> is located at the end of the first part to the right of 0 and is one-fourth of the way from 0 to 1.</i></p>


**Part 4: Properties and conventions of the number line (~15 minutes)**

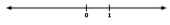
<b><i>Goals</i></b>	<b><i>Instructional sequence</i></b>	<b><i>Supplemental</i></b>
<ul style="list-style-type: none"> <li>Participants will begin to consider key mathematical features of the number line.</li> </ul>	<ol style="list-style-type: none"> <li>Introduce Part 4 and discuss the idea that representations have both properties and conventions associated with their construction and interpretation.</li> <li>Have participants brainstorm properties and conventions of the number line.</li> <li><i>Optional:</i> Watch video examples of other teachers' ideas.</li> </ol>	<ul style="list-style-type: none"> <li>Video A (00:15): Going on forever</li> <li>Video B (00:43): Special role of 0</li> <li>Video C (00:09): The unit interval</li> <li>Video D (00:59): Which numbers can be represented</li> <li>Video E (01:27): Distance and order</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce the Part 4: Representations, such as the number line, have both properties and conventions associated with their construction and interpretation. When placing numbers on the number line in Part 3, participants drew upon properties and conventions (perhaps implicitly) as they placed numbers on the number line. In teaching, it is important to be able to make the properties and conventions of representations explicit to students because it helps them interpret and use representations. It is also mathematics content. In this part, participants will begin to articulate key properties and conventions of the number line.</p>	<p><i>Consider having participants share the examples of number lines that they collected/ thought about in the CCA from session 3. These examples can be used to precipitate the conversation about the properties of number lines. For instance, they often don't show negatives, don't have arrows, many will show just whole numbers.</i></p>
<p>2. Have participants work with a partner to brainstorm: <i>What are some key mathematical properties and conventions of the number line?</i> Ask pairs to record their ideas because they will be revising and adding to their list throughout the session.</p> <p>Although you might elicit a few ideas in whole group, <u>do not engage in a detailed discussion at this point</u>. There will be a whole-group discussion of the list after watching the video in Part 5.</p>	<p><i>In the next part, Dr. Ball will introduce a list of key properties and conventions of the number line. These elements are shown to the right.</i></p>
<p>3. <i>Videos A – E</i> provide examples of other teachers' ideas about key mathematical features of the number line. As time permits, have participants watch and discuss some or all of the clips.</p> <ul style="list-style-type: none"> <li><i>Video A:</i> Going on forever</li> <li><i>Video B:</i> Special role of 0</li> <li><i>Video C:</i> The unit interval</li> <li><i>Video D:</i> Which numbers can be represented</li> <li><i>Video E:</i> Distance and order</li> </ul> <p>After watching each clip, have participants look over their list of number line properties to see if there is anything they would like to add or revise.</p>	<p><i>Depending on what you observed in the discussions, you might show a particular clip in whole group to draw out properties and conventions that have not been brought up.</i></p>

**Part 5: Summarizing key ideas about the number line (~20 minutes)**

<p><b>Goals</b></p> <ul style="list-style-type: none"> <li>Participants will be able to identify and use key properties and conventions of the number line as well as additional observations about the line.</li> </ul>	<p><b>Instructional sequence</b></p> <ol style="list-style-type: none"> <li>Introduce Part 1 and watch the video in which Dr. Ball introduces key properties and conventions of the number line.</li> <li>Compare the key elements with lists that were brainstormed in Part 4.</li> <li>Have participants read the slide, <i>Other observations about the number line</i>, and watch and discuss the associated videos as appropriate.</li> </ol>	<p><b>Resources</b></p> <ul style="list-style-type: none"> <li>Video A (05:26): Important properties and conventions of the number line</li> <li>Video B (01:20): Infinitely many equivalent forms of a number</li> <li>Video C (01:12): All integers can be written as a fraction</li> <li>Video D (00:48): The rational numbers are dense on the line</li> <li>Video E (00:45): Additive inverse</li> <li>Video F (00:38): The real line</li> <li>Video G (01:41): A mathematical object</li> <li>Handout: Summarizing key ideas about the number line</li> </ul> <p><b>Supplemental</b></p> <ul style="list-style-type: none"> <li>Math notes: Properties and conventions of the number line</li> </ul>
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Detailed description of activity	Comments & other resources
<p>1. Have participants watch <i>Video A</i> in which Dr. Ball introduces a set of key properties and conventions of the number line. Tell participants that they will be comparing this list to the ideas they brainstormed with their partner.</p> <div data-bbox="625 836 1008 1128" style="border: 1px solid black; padding: 5px;"> <p style="text-align: center;"><b>Key properties and conventions of the number line</b></p>  <p><small>The horizontally displayed number line has the following key properties and conventions:</small></p> <ul style="list-style-type: none"> <li>• <b>Locating Numbers:</b> The whole numbers are located on the number line by starting at 0 and repeatedly measuring the unit distance to the right. A fraction is similarly located on the number line by partitioning the unit interval into parts of equal length where the number of parts corresponds to the denominator. A fraction can be located by repeatedly measuring the length of one of these parts.</li> <li>• <b>Two infinite directions:</b> The number line has two directions and extends infinitely in both.</li> <li>• <b>Increase to the right:</b> Numbers are represented by points on the line. Numbers increase in value as you move to the right. In other words, <math>y &gt; x</math> if <math>y</math> is to the right of <math>x</math>.</li> <li>• <b>Positive and negative:</b> Once the number 0 is located, points to the right of 0 are "positive" numbers and points to the left of 0 are "negative" numbers.</li> <li>• <b>Unit interval:</b> The interval from 0 to 1 is defined to be the unit interval and its length is the unit distance.</li> <li>• <b>Labeling Points:</b> The number corresponding to a point is determined by its distance from 0 in relation to the unit distance and its direction from 0.</li> <li>• <b>Symmetry:</b> The number line can be reflected through 0. This reflection maps each number <math>x</math> to its additive inverse (opposite), <math>-x</math>.</li> </ul> <p><small>This work is licensed under a Creative Commons Attribution-NonCommercial-4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach in School of Education at University of Michigan. All Rights Reserved. 4</small></p> </div>	<p><i>You might point out that unpacking the number line representation is an example of a kind of teaching work that could be done for other representations for fractions, as well as representations more generally.</i></p> <p><i>It may help to pause the video to ensure that participants understand a particular idea that has been raised. You could: see if participants can restate a idea or provide an example of the idea; quickly survey the group to see if participants had recorded a particular idea in their lists and if so how they named that idea.</i></p> <p><b>CCSSM Link:</b> <i>The ideas in this module about how to locate and name fractions on the number line closely mirror the Grade 3 standards related to understanding and representing fractions on the number line (3.NF.2). CCSSM states:</i></p> <ul style="list-style-type: none"> <li><i>Represent a fraction <math>1/b</math> on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into <math>b</math> equal parts. Recognize that each part has size <math>1/b</math> and that the endpoint of the part based at 0 locates the number <math>1/b</math> on the number line.</i></li> <li><i>Represent a fraction <math>a/b</math> on a number line diagram by marking off a lengths <math>1/b</math> from 0. Recognize that the resulting interval has size <math>a/b</math> and that its endpoint locates the number <math>a/b</math> on the number line.</i></li> </ul>

Detailed description of activity	Comments & other resources
<p>2. Have participants compare the list of elements introduced by Dr. Ball with the list(s) brainstormed in the previous part of the session.</p> <p>After a few minutes, discuss in whole group:</p> <ul style="list-style-type: none"> <li>• How does this list compare to the one you made?</li> <li>• Is there anything you'd like to add or revise?</li> <li>• How do the ideas on the list connect to the work with the number line in the video?</li> </ul>	<p><i>Participants may comment on the appropriateness of sharing these ideas with students in early grades. In this case, you can point out that it is important for teachers to have these types of understandings of representations, even though they may not discuss them explicitly with students. Such understandings help teachers more skillfully use the representation with students, foreshadow key ideas, and elicit selected ideas with students at strategic times.</i></p>
<p>3. Have participants read the slide, <i>Other observations about the number line.</i></p> <p>Watch and discuss associated video associated with points on the slide as appropriate.</p> <ul style="list-style-type: none"> <li>• Video B (01:20): Infinitely many equivalent forms of a number</li> <li>• Video C (00:12): All integers can be written as a fraction</li> <li>• Video D (00:48): The rational numbers are dense on the line</li> <li>• Video E (00:45): Additive inverse</li> <li>• Video F (00:37): The real line</li> <li>• Video G (01:41): A mathematical object</li> </ul> <p>Distribute the <i>Handout: Summarizing key ideas about the number line</i> and the <i>Math Notes</i> document if it is useful in your context.</p>	<div data-bbox="625 570 1010 857" style="border: 1px solid black; padding: 5px;"> <p style="text-align: center;"><b>Other observations about the number line</b></p>  <ul style="list-style-type: none"> <li>• Every number can be written in infinitely many equivalent forms.</li> <li>• All integer points can be written as fractions (e.g., the integer 2 can be written as <math>\frac{2}{1}</math> or <math>\frac{4}{2}</math>), but not all fractions represent integers (e.g., <math>\frac{1}{2}</math> cannot be written as an integer).</li> <li>• Between any two points on the number line there is always a rational number (a number that can be written as a fraction). This is what we mean when we say that the rational numbers are dense in the line.</li> <li>• Any number, <math>x</math>, and its additive inverse, <math>-x</math>, are the same distance from 0. For example, <math>-2</math> and <math>2</math> are both at distance 2 from 0.</li> <li>• The numbers represented by points on the number line are called the real numbers and so the number line is also called the "real line." The real numbers include the rational numbers, but not all real numbers are rational; those that are not are called <i>irrational numbers</i> (e.g., square root of 2 and pi can be shown to be irrational).</li> <li>• The number line is an important mathematical object. Some features of the number line are accessible to children in the early elementary grades.</li> </ul> <p style="font-size: small;">This work is licensed under a Creative Commons Attribution-NonCommercial-4.0 International License: <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu 5</p> </div>

**Part 6: Analyzing students' errors when labeling marked points on the number line (~25 minutes)**

**Goals**

- Participants will be able to apply their ideas on the definition of fraction, number lines, and interpreting student thinking about representations to anticipate, analyze, and develop strategies for responding to student thinking about labeling locations (points) on a number line.
- Participants will be able to identify 5 common errors students make when labeling locations (points) on a number line.

**Instructional sequence**

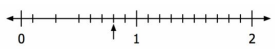
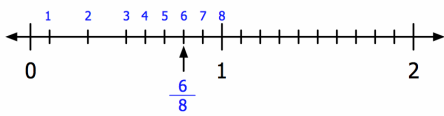
1. Introduce Part 6 and have participants read the number line task in the slide.
2. For each number displayed in the viewer, participants should:
  - Explain how a student might have arrived at that answer.
  - Click on a student answer and read a common student explanation for that response.
  - Consider how you might respond to a student who gave that answer.

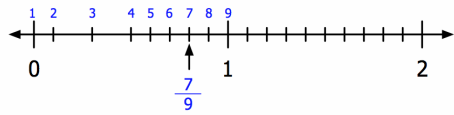
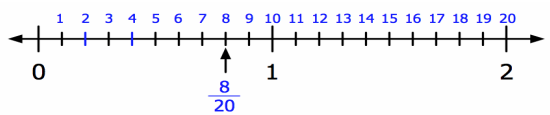
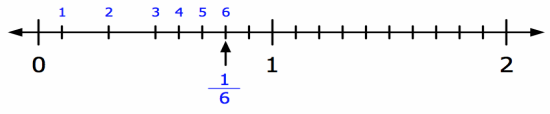
**Resources**

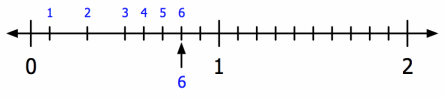
- Handout: Number line task
- Handout: A working definition of a fraction

**Supplemental**

- Math notes: The strategies underlying students' errors when identifying points on the number line

Detailed description of activity	Comments & other resources
<p>1. Introduce Part 6: In addition to teaching students to locate numbers on a number line, an important goal of elementary mathematics instruction is to help students learn to label points marked on the line. In this part of the session, participants explore patterns in students' thinking about the number line and consider how their errors relate to the key properties of the number line identified in this session.</p> <p>Have participants read the task on the slide. In this task, the interval from 0 to 1 is divided into <u>unequal</u> parts, and students are asked to label the point indicated by the arrow. The task is designed to find out whether students recognize the importance of equal parts when naming a point on the number line.</p> <p>Ask participants to anticipate the types of responses that students would give. Then, reveal five common answers given by fifth graders.</p> <div data-bbox="661 941 1039 1226" data-label="Complex-Block"> <p style="text-align: center;"><b>Labeling locations on the number line</b></p> <p style="text-align: center;">What value should be written where the arrow is pointing?</p>  <p style="font-size: small; text-align: center;">Shaughnessy, M. H. (2009). Students' Flexible Use of Multiple Representations for Rational Number: Decimals, Fractions, Parts of Area, and Number Lines. Unpublished doctoral dissertation, University of California, Berkeley. This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu</p> </div>	<p><b>CCSSM Link:</b> <i>Understanding a fraction as a number on the number line and representing fractions on a number line is a Grade 3 standard (3.NF.2).</i></p> <p><b>Common responses:</b></p> <div data-bbox="1071 893 1848 1291" data-label="Complex-Block"> <p>6/8</p>  <p><i>There's eight spaces and this is the end of the sixth space.</i></p> <p><b>Response indicates:</b> The student is focusing on numbers of tick marks (or parts) rather than on distances.</p> <p><b>Key points to raise:</b> When we label a fraction on the number line, we need to partition the whole (the unit) into parts of equal length.</p> </div>

Detailed description of activity	Comments & other resources	
<p>The numbers displayed in the viewer (6/8; 7/9; 8/20; 1/6; and 6) are common incorrect student responses to this task from fifth grade students. For each response, have participants work with a partner to complete the following tasks:</p> <ol style="list-style-type: none"> <li>1) Explain how a student might have arrived at that answer.</li> <li>2) Consider how you might respond to a student who gave that response, in particular: <ul style="list-style-type: none"> <li>• What question(s) would you ask to learn more about student's thinking?</li> <li>• What key point(s) about the number line might you raise with that student?</li> </ul> </li> </ol> <p>When most pairs have finished, chose one or two of the incorrect responses to discuss ideas for how they would respond to a student who gave that response (question #3 above).</p> <p>Distribute the <i>Math Notes</i> document if it is useful in your context.</p>	<p>7/9</p>	 <p><i>There are nine lines and the arrow points to the seventh one.</i></p> <p><i>Response indicates:</i> The student is focusing on numbers of tick marks (or parts) rather than on distances.</p> <p><i>Key points to raise:</i> When we label a fraction on the number line, we need to partition the whole (the unit) into parts of equal length.</p>
	<p>8/20</p>	 <p><i>It's ten from 0 to 1 so it has to be another ten from 1 to 2 so that would make twenty, then I counted from 0 to the arrow to get eight.</i></p> <p><i>Response indicates:</i> The student has redefined the unit interval. They are treating the entire interval shown as the unit interval. Given the unit distance that they are using, they are coordinating the distance of the point from 0 in relation to the unit distance. They are doing this by partitioning the unit interval into parts of equal length.</p> <p><i>Key points to raise:</i> The interval from 0 to 1 is the whole (the unit).</p>
	<p>1/6</p>	 <p><i>One, two, three, four, five, six and then I saw the one [referring to the 1 marked on the number line] and put the one there [referring to the 1 in the numerator].</i></p> <p><i>Response indicates:</i> The student is counting discrete quantities (parts or tick marks) in ways that do not attempt to coordinate the number of discrete quantities from 0 to the target point with the number of discrete quantities from 0 to 1.</p> <p><i>Key points to raise:</i> The importance of the whole (the unit).</p>

Detailed description of activity	Comments & other resources
	<div style="display: flex; align-items: flex-start;"> <div style="margin-right: 10px;">6</div> <div style="text-align: center;">  </div> </div> <p><i>It is the sixth one.</i></p> <p><i>Response indicates:</i> The student is counting discrete quantities (parts or tick marks) in ways that do not attempt to coordinate the distance of the point from 0 in relation to the unit distance.</p> <p><i>Key points to raise:</i> The importance of the whole (the unit).</p> <p><i>Encourage participants to make use of the working definition of fractions and the properties and conventions of the number line when thinking about how they would respond to each student.</i></p>



**Part 7: Wrap up (~5 minutes)**

<b><u>Goals</u></b>	<b><u>Instructional sequence</u></b>	<b><u>Resources</u></b>
<ul style="list-style-type: none"> <li>Participants will understand ways of connecting the session content to their classroom.</li> </ul>	<ol style="list-style-type: none"> <li>Summarize the work of the session.</li> <li>Distribute the Classroom Connection Activities handout.</li> </ol>	

Detailed description of activity	Comments & other resources
<p>1. Summarize the work of the session. In this session, participants:</p> <ul style="list-style-type: none"> <li>Analyzed a student’s use of the number line</li> <li>Explored properties and conventions of the number line</li> <li>Analyzed students’ errors when using the number line</li> </ul> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #1a3d4d; color: white; padding: 2px;"><b>Summary</b></p> <p>In this session, you:</p> <ul style="list-style-type: none"> <li>Analyzed a student’s use of the number line</li> <li>Explored properties and conventions of the number line</li> <li>Analyzed students’ errors when using the number line</li> </ul> <p style="font-size: 8px; margin-top: 5px;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu 7</p> </div>	
<p>2. Distribute the handout you customized with selected Classroom Connection Activities and accompanying documents described below.</p> <p><b>Required:</b></p> <ul style="list-style-type: none"> <li>Analyze curriculum materials to identify treatment and features of number lines.</li> <li>Analyze student work samples and anticipate the reasoning that student may have used to locate numbers on the number line.</li> </ul> <p><b>Optional:</b></p> <ul style="list-style-type: none"> <li>Read an excerpt on the number line as a geometric representation of number from the National Research Council (2001) report, <i>Adding It Up: Helping Children Learn Mathematics</i>.</li> </ul>	<p><i>It may be helpful to ask participants to upload their responses or materials related to the Classroom Connection Activities so that you can review what participants have been thinking and trying prior to the next sessions. In addition, uploading responses would allow participants to easily share their ideas with each other.</i></p>