National Research Council. (2001). Strategic competence. In J. Kilpatrick, J. Swafford, and B. Findell (Eds.),*Adding it up: Helping children learn mathematics* (pp. 124-129). Washington, DC: National Academies Press. Reprinted with permission by the National Academy of Sciences, courtesy of the National Academies Press, Washington, D.C.

with understanding can modify or adapt procedures to make them easier to use. For example, students with limited understanding of addition would ordinarily need paper and pencil to add 598 and 647. Students with more understanding would recognize that 598 is only 2 less than 600, so they might add 600 and 647 and then subtract 2 from that sum.[20]

Strategic Competence

Strategic competence refers to the ability to formulate mathematical problems, represent them, and solve them.

Strategic competence refers to the ability to formulate mathematical problems, represent them, and solve them. This strand is similar to what has been called problem solving and problem formulation in the literature of mathematics education and cognitive science, and mathematical problem solving, in particular, has been studied extensively.[21]

Although in school, students are often presented with clearly specified problems to solve, outside of school they encounter situations in which part of the difficulty is to figure out exactly what the problem is. Then they need to formulate the problem so that they can use mathematics to solve it. Consequently, they are likely to need experience and practice in problem formulating as well as in problem solving. They should know a variety of solution strategies as well as which strategies might be useful for solving a specific problem. For example, sixth graders might be asked to pose a problem on the topic of the school cafeteria.[22] Some might ask whether the lunches are too expensive or what the most and least favorite lunches are. Others might ask how many trays are used or how many cartons of milk are sold. Still others might ask how the layout of the cafeteria might be improved.

With a formulated problem in hand, the student's first step in solving it is to represent it mathematically in some fashion, whether numerically, symbolically, verbally, or graphically. Fifth graders solving problems about getting from home to school might describe verbally the route they take or draw a scale map of the neighborhood. Representing a problem situation requires, first, that the student build a mental image of its essential components. Becoming strategically competent involves an avoidance of "number grabbing" methods (in which the student selects numbers and prepares to perform arithmetic operations on them)[23] in favor of methods that generate problem models (in which the student constructs a mental model of the variables and relations described in the problem). To represent a problem accurately, students must first understand the situation, including its key features. They then need to generate a mathematical representation of the problem that captures the core mathematical elements and ignores the irrelevant features. This step may be facilitated by making a drawing, writing an equation, or creating some other tangible representation. Consider the following two-step problem:

> *At ARCO, gas sells for $1.13 per gallon.*
> *This is 5 cents less per gallon than gas at Chevron.*
> *How much does 5 gallons of gas cost at Chevron?*

In a common superficial method for representing this problem, students focus on the numbers in the problem and use so-called keywords to cue appropriate arithmetic operations.[24] For example, the quantities *$1.83* and *5 cents* are followed by the keyword *less*, suggesting that the student should subtract 5 cents from $1.13 to get $1.08. Then the keywords *how much* and *5 gallons* suggest that 5 should be multiplied by the result, yielding $5.40.

In contrast, a more proficient approach is to construct a problem model—that is, a mental model of the situation described in the problem. A problem model is not a visual picture per se; rather, it is any form of mental representation that maintains the structural relations among the variables in the problem. One way to understand the first two sentences, for example, might be for a student to envision a number line and locate each cost per gallon on it to solve the problem.

In building a problem model, students need to be alert to the quantities in the problem. It is particularly important that students represent the quantities mentally, distinguishing what is known from what is to be found. Analyses of students' eye fixations reveal that successful solvers of the two-step problem above are likely to focus on terms such as *ARCO*, *Chevron*, and *this*, the principal known and unknown quantities in the problem. Less successful problem solvers tend to focus on specific numbers and keywords such as *$1.13*, *5 cents*, *less*, and *5 gallons* rather than the relationships among the quantities.[25]

Not only do students need to be able to build representations of individual situations, but they also need to see that some representations share common mathematical structures. Novice problem solvers are inclined to notice similarities in surface features of problems, such as the characters or scenarios described in the problem. More expert problem solvers focus more on the structural relationships within problems, relationships that provide the clues for how problems might be solved.[26] For example, one problem might ask students to determine how many different stacks of five blocks can be made using red and green blocks, and another might ask how many different ways hamburgers can be ordered with or without each of the following:

catsup, onions, pickles, lettuce, and tomato. Novices would see these problems as unrelated; experts would see both as involving five choices between two things: red and green, or with and without.[27]

In becoming proficient problem solvers, students learn how to form mental representations of problems, detect mathematical relationships, and devise novel solution methods when needed. A fundamental characteristic needed throughout the problem-solving process is flexibility. Flexibility develops through the broadening of knowledge required for solving nonroutine problems rather than just routine problems.

Routine problems are problems that the learner knows how to solve based on past experience.[28] When confronted with a routine problem, the learner knows a correct solution method and is able to apply it. Routine problems require reproductive thinking; the learner needs only to reproduce and apply a known solution procedure. For example, finding the product of 567 and 46 is a routine problem for most adults because they know what to do and how to do it.

In contrast, nonroutine problems are problems for which the learner does not immediately know a usable solution method. Nonroutine problems require productive thinking because the learner needs to invent a way to understand and solve the problem. For example, for most adults a nonroutine problem of the sort often found in newspaper or magazine puzzle columns is the following:

> *A cycle shop has a total of 36 bicycles and tricycles in stock.*
> *Collectively there are 80 wheels.*
> *How many bikes and how many tricycles are there?*

One solution approach is to reason that all 36 have at least two wheels for a total of $36 \times 2 = 72$ wheels. Since there are 80 wheels in all, the eight additional wheels $(80 - 72)$ must belong to 8 tricycles. So there are $36 - 8 = 28$ bikes.

A less sophisticated approach would be to "guess and check": If there were 20 bikes and 16 tricycles, that would give $(20 \times 2) + (16 \times 3) = 88$ wheels, which is too many. Reducing the number of tricycles, a guess of 24 bikes and 12 tricycles gives $(24 \times 2) + (12 \times 3) = 84$ wheels—still too many. Another reduction of the number of tricycles by 4 gives 28 bikes, 8 tricycles, and the 80 wheels needed.

A more sophisticated, algebraic approach would be to let b be the number of bikes and t the number of tricycles. Then $b + t = 36$ and $2b + 3t = 80$. The solution to this system of equations also yields 28 bikes and 8 tricycles.

A student with strategic competence could not only come up with several approaches to a nonroutine problem such as this one but could also choose flexibly among reasoning, guess-and-check, algebraic, or other methods to suit the demands presented by the problem and the situation in which it was posed.

Flexibility of approach is the major cognitive requirement for solving nonroutine problems. It can be seen when a method is created or adjusted to fit the requirements of a novel situation, such as being able to use general principles about proportions to determine the best buy. For example, when the choice is between a 4-ounce can of peanuts for 45 cents and a 10-ounce can for 90 cents, most people use a ratio strategy: the larger can costs twice as much as the smaller can but contains more than twice as many ounces, so it is a better buy. When the choice is between a 14-ounce jar of sauce for 79 cents and an 18-ounce jar for 81 cents, most people use a difference strategy: the larger jar costs just 2 cents more but gets you 4 more ounces, so it is the better buy. When the choice is between a 3-ounce bag of sunflower seeds for 30 cents and a 4-ounce bag for 44 cents, the most common strategy is unit-cost: The smaller bag costs 10 cents per ounce, whereas the larger costs 11 cents per ounce, so the smaller one is the better buy.

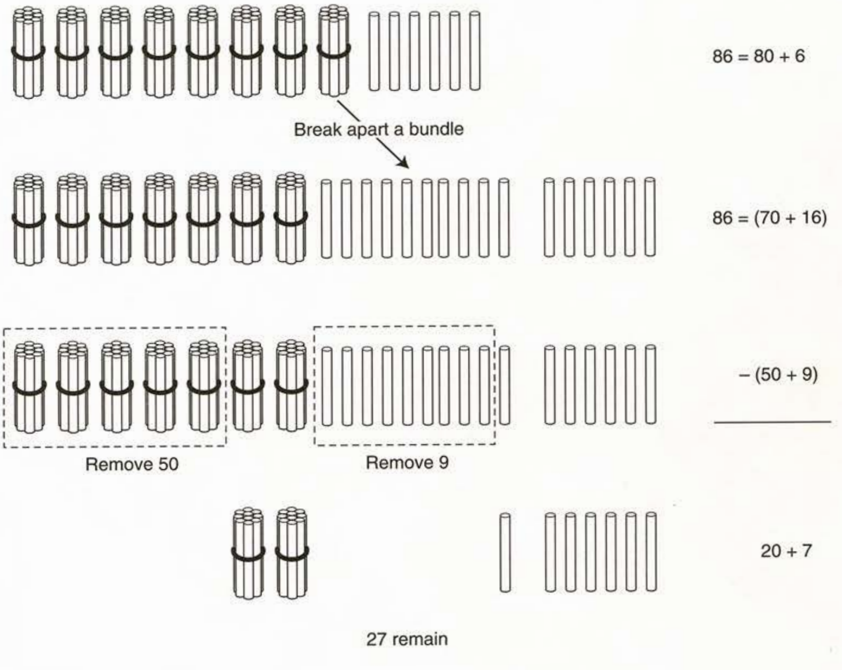
There are mutually supportive relations between strategic competence and both conceptual understanding and procedural fluency,

There are mutually supportive relations between strategic competence and both conceptual understanding and procedural fluency, as the various approaches to the cycle shop problem illustrate. The development of strategies for solving nonroutine problems depends on understanding the quantities involved in the problems and their relationships as well as on fluency in solving routine problems. Similarly, developing competence in solving nonroutine problems provides a context and motivation for learning to solve routine problems and for understanding concepts such as *given*, *unknown*, *condition*, and *solution*.

Strategic competence comes into play at every step in developing procedural fluency in computation. As students learn how to carry out an operation such as two-digit subtraction (for example, 86 – 59), they typically progress from conceptually transparent and effortful procedures to compact and more efficient ones (as discussed in detail in chapter 6). For example, an initial procedure for 86 – 59 might be to use bundles of sticks (see Box 4-3). A compact procedure involves applying a written numerical algorithm that carries out the same steps without the bundles of sticks. Part of developing strategic competence involves learning to replace by more concise and efficient procedures those cumbersome procedures that might at first have been helpful in understanding the operation.

Box 4-3

Subtraction Using Sticks: Modeling 86 – 59 = ?



Begin with 8 bundles of 10 sticks along with 6 individual sticks. Because you cannot take away 9 individual sticks, open one bundle, creating 7 bundles of 10 sticks and 16 individual sticks. Take away 5 of the bundles (corresponding to subtracting 50), and take away 9 individual sticks (corresponding to subtracting 9). The number of remaining sticks—2 bundles and 7 individual sticks, or 27—is the answer.

Students develop procedural fluency as they use their strategic competence to choose among effective procedures. They also learn that solving challenging mathematics problems depends on the ability to carry out procedures readily and, conversely, that problem-solving experience helps them acquire new concepts and skills. Interestingly, very young children use a variety of strategies to solve problems and will tend to select strategies that are well suited to particular problems.[29] They thereby show the rudiments of adaptive reasoning, the next strand to be discussed.

Adaptive Reasoning

Adaptive reasoning refers to the capacity to think logically about the relationships among concepts and situations. Such reasoning is correct and valid, stems from careful consideration of alternatives, and includes knowledge of how to justify the conclusions. In mathematics, adaptive reasoning is the glue that holds everything together, the lodestar that guides learning. One uses it to navigate through the many facts, procedures, concepts, and solution methods and to see that they all fit together in some way, that they make sense. In mathematics, deductive reasoning is used to settle disputes and disagreements. Answers are right because they follow from some agreed-upon assumptions through series of logical steps. Students who disagree about a mathematical answer need not rely on checking with the teacher, collecting opinions from their classmates, or gathering data from outside the classroom. In principle, they need only check that their reasoning is valid.

Adaptive reasoning refers to the capacity to think logically about the relationships among concepts and situations.

Many conceptions of mathematical reasoning have been confined to formal proof and other forms of deductive reasoning. Our notion of adaptive reasoning is much broader, including not only informal explanation and justification but also intuitive and inductive reasoning based on pattern, analogy, and metaphor. As one researcher put it, "The human ability to find analogical correspondences is a powerful reasoning mechanism."[30] Analogical reasoning, metaphors, and mental and physical representations are "tools to think with," often serving as sources of hypotheses, sources of problem-solving operations and techniques, and aids to learning and transfer.[31]

Some researchers have concluded that children's reasoning ability is quite limited until they are about 12 years old.[32] Yet when asked to talk about how they arrived at their solutions to problems, children as young as 4 and 5 display evidence of encoding and inference and are resistant to counter suggestion.[33] With the help of representation-building experiences, children can demonstrate sophisticated reasoning abilities. After working in pairs and