

***Description of the Session 9: Investigating strategies for comparing fractions***

In this session, participants will first have a conversation about a CCA from last session focused on their most recent cycle of learning from the use of public recording space. In this session, participants explore different strategies for comparing fractions as a way to reconnect with mathematical ideas developed earlier in the module, to consider and classify how students compare fractions, and to work on teaching practices related to the use of representations. Participants explore strategies for representing and comparing fractions and then draw on insights from this work to interpret students' comparison of fractions. Lastly, participants examine equivalence, the ways in which equivalent fractions may be represented, and the limitations of particular representations of equivalence.

***Activities and goals of the session***

<b>Activities</b>	<b>Times</b>	<b>Corresponding parts of the session</b>	<b>Goals</b>
Conversation about a CCA from the last session	30 minutes		<ul style="list-style-type: none"> <li>• Participants will be able to use a checklist to help them analyze records from their use of public recording space.</li> </ul>
I. Preview	5 minutes	Part 1	<ul style="list-style-type: none"> <li>• Participants will be oriented to the work of the session.</li> </ul>
II. Exploring strategies and representations for comparing fractions	55 minutes	Parts 2, 3 & 4	<ul style="list-style-type: none"> <li>• Participants will be able to:               <ul style="list-style-type: none"> <li>○ talk about the advantages/disadvantages of particular representations for comparing particular pairs of fractions;</li> <li>○ explain four different strategies to compare fractions; and</li> <li>○ identify fraction comparison strategies shown in student work, evaluate whether these strategies are valid, and generate follow-up questions to ask children about their strategies.</li> </ul> </li> </ul>
III. Showing and explaining equivalence	25 minutes	Part 5	<ul style="list-style-type: none"> <li>• Participants will be able to:               <ul style="list-style-type: none"> <li>○ explain what it means for two fractions to be equivalent with respect to particular representations;</li> <li>○ show and explain how to generate equivalent fractions; and</li> <li>○ explain the limitations of representing equivalence using particular representations.</li> </ul> </li> </ul>
IV. Wrap up	5 minutes	Part 6	<ul style="list-style-type: none"> <li>• Participants will understand the Classroom Connection Activities.</li> </ul>

***Classroom Connection Activities***

Required	Optional
Type of task: Reflective paper Description: Reflection on the use of public recording space in classroom	Type of task: Mathematics Description: Explaining solutions to a fraction-of-an-area task involving multiple wholes
Type of task: Teaching practice Description: Narrating fraction comparison tasks using area models and set models	Type of task: Teaching practice Description: Excerpt on strategic competence from National Research Council's (2001) <i>Adding it Up</i>

***Preparing for the session***

- Make copies as needed:
  - *Resources:* Handout: Public recording space checklist (see Session 8); Handout: Fraction comparison problems (Part 2); Handout: Student work 1-6 (Part 4); Handout: Textbook examples of fraction equivalence (Part 5)
  - *Supplements:* Math notes: Strategies for comparing fractions (Part 3); Math notes: Methods for generating and explaining equivalent fractions (Part 5)
- Customize the Classroom Connection Activities and make copies as needed
- Test technical setups (Internet connection, speakers, projector)

***Developing a culture for professional work on mathematics teaching (ongoing work of the facilitator throughout the module)***

1. Encourage participation: talking in whole-group discussions; rehearsing teaching practices; coming up to the board as appropriate.
2. Develop habits of speaking and listening: speaking so that others can hear; responding to others' ideas, statements, questions, and teaching practices.
3. Develop norms for talking about teaching practice: close and detailed talk about the practice of teaching; supporting claims with specific examples and evidence; curiosity and interest in other people's thinking; serious engagement with problems of mathematics learning and teaching.
4. Develop norms for mathematical work:
  - a) Reasoning: explaining in detail; probing reasons, ideas, and justifications; expectation that justification is part of the work; attending to others' ideas with interest and respect.
  - b) Representing: building correspondences and making sense of representations, as well as the ways others construct and explain them.
  - c) Carefully using mathematical language.

5. Help participants make connections among module content and develop the sense that this module will be useful in helping them improve their mathematics teaching, their knowledge of mathematics, their understanding of student thinking, and their ability to learning from their own teaching.
6. Help participants understand connections between module content and the Common Core Standards for School Mathematics.

*Scope of the module (focal content of this session in **bold**)*

Mathematics	Student thinking	Teaching practice	Learning from practice
<ul style="list-style-type: none"> <li>• representing fractions</li> <li>• defining fractions</li> <li>• <b>using and explaining methods and representations for comparing fractions</b></li> <li>• <b>understanding how equivalence (of fractions) can be represented and used</b></li> </ul>	<ul style="list-style-type: none"> <li>• identifying and analyzing student conceptions, explanations, and representations of fractions</li> <li>• <b>identifying and analyzing student strategies for comparing fractions</b></li> </ul>	<ul style="list-style-type: none"> <li>• <b>selecting and generating representations</b></li> <li>• connecting representations</li> <li>• narrating the process of representing</li> <li>• supporting students in narrating the use of a representation</li> <li>• recording contributions and emerging mathematical ideas</li> </ul>	<ul style="list-style-type: none"> <li>• studying public recording space to learn from practice</li> <li>• using a conceptual framework to guide the planning, use, and analysis of public recording space</li> </ul>

***Conversation about a Classroom Connection Activity from the last session (~30 minutes)***

<b><u>Goals</u></b>	<b><u>Instructional sequence</u></b>	<b><u>Resources</u></b>
<ul style="list-style-type: none"> <li>Participants will be able to use a checklist to help them analyze records from their use of public recording space.</li> </ul>	<ol style="list-style-type: none"> <li>Discuss most recent cycle of learning from the use of public recording space with a partner.</li> </ol>	<ul style="list-style-type: none"> <li>Public recording space checklist (see Session 8)</li> </ul>

Detailed description of activity	Comments & other resources
<ol style="list-style-type: none"> <li>Have participants work in pairs to discuss their most recent cycle of learning from the use of public recording space. Each partner should share the context of her/his work and then use the public recording space checklist to analyze the plans, images, and reflections that they bring to the session.</li> </ol>	<p><i>If time is limited, have participants work in the same groups that they worked in during Session 8, otherwise it might be generative to have them work with new partners.</i></p> <p><i>Consider posing the following questions to participants:</i></p> <ul style="list-style-type: none"> <li><i>What ideas in the checklist stood out to you as important to think more about?</i></li> <li><i>What ways of documenting your plans or images of the board have been effective/ineffective in supporting your engagement in this process?</i></li> <li><i>Which records seem particularly useful in reflecting on your practice? What additional documentation might help?</i></li> </ul>

**Part 1: Preview (~5 minutes)**

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
<ul style="list-style-type: none"> <li>Participants will be oriented to the work of the session.</li> </ul>	1. Introduce the session and watch the introductory video.	<ul style="list-style-type: none"> <li>Video (02:56): Overview</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce session: Session 9 explores different strategies for comparing fractions as a way to:</p> <ul style="list-style-type: none"> <li>Unpack fundamental mathematical ideas;</li> <li>Consider and classify how students compare fractions; and</li> <li>Work on teaching practices related to the use of representations.</li> </ul> <p>The session also includes a focus on showing and explaining equivalence of fractions.</p> <p>Have the participants watch the <i>video</i> in which Dr. Ball presents an overview of the content that will be covered in Session 9.</p> <div data-bbox="583 532 961 820" style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #2c4e64; color: white; padding: 2px;">Overview of Session 9</p> <ul style="list-style-type: none"> <li>Exploring strategies and representations for comparing fractions</li> <li>Showing and explaining equivalence of fractions</li> </ul> <p style="font-size: 8px; margin-top: 5px;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu</p> </div>	

**Part 2: Analyzing fraction-comparison problems (~15 minutes)**

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
<ul style="list-style-type: none"> <li>Participants will begin to be able to identify strategies and representations for comparing fractions.</li> <li>Participants will begin to be able to talk about the advantages/disadvantages of particular representations for comparing particular pairs of fractions.</li> </ul>	<ol style="list-style-type: none"> <li>1. Introduce Part 2 and have participants analyze the comparison problems using the focus questions.</li> </ol>	<ul style="list-style-type: none"> <li>Handout: Fraction comparison problems</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce Part 2: Carefully analyzing the mathematical possibilities of tasks provides an important foundation for teaching. It helps teachers hear the mathematics in what students say, guides selection and modification of problems, and yields mathematical considerations that can be coordinated with a teacher’s knowledge of student thinking and instructional goals to make decisions. This part focuses on analyzing a set of fraction-comparison problems to identify and compare possible solution strategies and representations.</p> <p>Have participants analyze the comparison problems on the slide in the viewer (see Handout: Exploring the possibilities of student math problems). Ask participants to:</p> <ul style="list-style-type: none"> <li>Identify different strategies that could be used to compare the fractions.</li> <li>Name the strategy they used.</li> <li>Try using different representations to explain their strategy, and think about the affordances and limitations of each representation.</li> </ul> <p>After participants have had time to work on these tasks in pairs, return to whole group to elicit ideas about some of the tasks.</p> <div data-bbox="583 883 961 1170" style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #2c4e64; color: white; padding: 2px;"><b>Fraction comparison problems</b></p> <p style="text-align: center;">Which fraction is larger?</p> <p>a. <math>\frac{4}{3}</math> or <math>\frac{14}{15}</math>      b. <math>\frac{3}{4}</math> or <math>\frac{14}{15}</math></p> <p>c. <math>\frac{1}{4}</math> or <math>\frac{5}{8}</math>      d. <math>\frac{2}{5}</math> or <math>\frac{2}{3}</math></p> <p style="font-size: 8px; text-align: center;">This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu 2</p> </div>	<p><i>The goal here is NOT to have participants anticipate student thinking about the problems, rather, the goal is to have participants consider the strategies that one could use and then think about the representations one could use to explain their strategy.</i></p> <p><i>The following are four strategies for comparing fractions by Van de Walle et al. (2009):</i></p> <ul style="list-style-type: none"> <li>More of the same-size part (common denominator)</li> <li>Same number of parts, but parts of different sizes (common numerator)</li> <li>More and less than a benchmark such as one-half or one whole</li> <li>Distance from a benchmark such as one-half or one whole (could be distances more than or less than)</li> </ul> <p><i>If one does not use a common denominator strategy, the "most likely" strategy for each example is:</i></p> <ul style="list-style-type: none"> <li>Comparison A: More and less than the benchmark of 1.</li> <li>Comparison B: Distance from 1</li> <li>Comparison C: More and less than the benchmark of <math>\frac{1}{2}</math></li> <li>Comparison D: Same number of parts, but parts of different sizes (common numerators)</li> </ul> <p><i>CCSSM Link: Comparing fractions with different numerators and denominators is a Grade 4 standard (4.NF.2).</i></p> <p><i>As students develop different strategies for comparing fractions, they draw upon and deepen their number sense. Developing different strategies for comparing fractions is also important because it supports students’ ability to solve fraction comparison problems efficiently and check the reasonableness of their solutions.</i></p>

**Part 3: Four strategies for comparing fractions (~20 minutes)**

<p><b><u>Goals</u></b></p> <ul style="list-style-type: none"> <li>Participants will understand and be able to explain four different strategies to compare fractions.</li> </ul>	<p><b><u>Instructional sequence</u></b></p> <ol style="list-style-type: none"> <li>Introduce Part 3 and describe four strategies for comparing fractions.</li> <li>Discuss common denominator strategy using the Image: Common denominator and watch selected videos (Videos A-C).</li> <li>Watch and discuss Video D.</li> <li>Watch and discuss Videos E – H.</li> </ol>	<p><b><u>Resources</u></b></p> <ul style="list-style-type: none"> <li>Image: Common denominator (1/4 and 5/8)</li> <li>Video A (01:13): Common denominators – Using the identity property</li> <li>Video B (01:11) Common denominators – Why is multiplication used</li> <li>Video C (01:00): Common denominators – Other parts of the process to consider</li> <li>Video D (01:55): Common numerator (2/3 and 2/5)</li> <li>Video E (00:35): More and less than one whole (4/3 and 14/15)</li> <li>Video F (00:22): Distance from one whole (3/4 and 14/15)</li> </ul> <p><b><u>Supplements</u></b></p> <ul style="list-style-type: none"> <li>Math notes: Strategies for comparing fractions</li> </ul>
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Detailed description of activity	Comments & other resources
<p>1. Introduce participants to four common strategies for comparing fractions that are drawn from a book by Van de Walle et al. (2009):</p> <ul style="list-style-type: none"> <li>More of the same-size parts (common denominator)</li> <li>Same number of parts, but parts of different sizes (common numerator)</li> <li>More and less than a benchmark such as one-half or one whole</li> <li>Distance from a benchmark such as one-half or one whole (could be distances more than or less than)</li> </ul> <div data-bbox="636 813 1018 1101" style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center;"><b>Comparison strategies</b></p> <ol style="list-style-type: none"> <li>More of the same-size parts (common denominator)</li> <li>Same number of parts but parts of different sizes (common numerator)</li> <li>More and less than one-half or one whole</li> <li>Distance from one-half or one whole (could be distance more than or distance less than)</li> </ol> <p style="text-align: right; font-size: small;">(Van de Walle, 2007)</p> <p style="font-size: x-small; text-align: center;">©2012MTEP • School of Education • University of Michigan • 734.202.4612 • <a href="http://dte.mtu.edu">dte.mtu.edu</a> All rights reserved. Please do not distribute or use without permission. 3</p> </div>	<p><i>Participants may have used some or all of four strategies analyzing the fractions comparison problems in part 2.</i></p>

Detailed description of activity	Comments & other resources
<p>2. Have the participants engage in unpacking the mathematics of the strategies starting with the example shown in the Image – Common denominator (1/4 and 5/8). Start by asking them why the method shown could be referred to as “more of the same-size parts”.</p> <p>Next ask participants to describe what they see happening mathematically in the strategy. Ask them to think about particular aspects of this strategy using <i>Videos A-C</i> and the discussion questions under each video.</p> <p>In Video A, a teacher describes reasoning that could be used to compare <math>\frac{4}{3}</math> and <math>\frac{14}{15}</math>. The teacher generates an equivalent fraction for <math>\frac{4}{3}</math> that has the same denominator as <math>\frac{14}{15}</math>. Have participants consider: Why does multiplying by <math>\frac{5}{5}</math> generate an equivalent fraction?</p> <p>In Video B, the teacher describes a situation where a student tried to generate a fraction equivalent to <math>\frac{5}{8}</math> by subtracting 2 from the numerator and the denominator. Have participants consider: Why wouldn't subtracting 2 from the numerator and from the denominator generate an equivalent fraction?</p> <p>In Video C, the teacher discusses the task: Compare <math>\frac{4}{3}</math> and <math>\frac{14}{15}</math>. She wonders why the student chose to multiply <math>\frac{4}{3}</math> by <math>\frac{5}{5}</math> rather than by another form of 1. Have participants consider: How do you think about which equivalent fraction to generate?</p> <p>If further unpacking seems necessary, refer participants to the Math notes document in the Supplements section.</p>	<p><i>Make clear to participants that the strategy used in image is not the only, and perhaps not the best, strategy for comparing the fractions. In this case we are considering the strategy because a student may use this strategy for this problem.</i></p> <p><i>Video A: Why does multiplying by <math>\frac{5}{5}</math> generate an equivalent fraction?</i></p> <p><i>Possible points to make:</i></p> <ul style="list-style-type: none"> <li><i>Multiplying by <math>\frac{5}{5}</math> is just like multiplying by 1. Multiplying by 1 yields a fraction equal to the one you start with. This illustrates the use of the identity property for multiplication.</i></li> <li><i>Multiplying by 5 in the denominator can be thought of as dividing each equal part of the original fraction into fifths. Multiplying by 5 in the numerator can be thought of as indicating five times as many of the new sized parts.</i></li> </ul> <p><i>Video B: Why wouldn't subtracting 2 from the numerator and from the denominator generate an equivalent fraction?</i></p> <p><i>Possible points to make:</i></p> <ul style="list-style-type: none"> <li><i>Students often think that adding (or subtracting) the same number to the numerator and denominator of a fraction yields an equivalent fraction. In this way of thinking, <math>(5 - 2)/(8 - 2)</math> yields a result that students may believe is equivalent to <math>\frac{5}{8}</math> because they are "doing the same thing to the bottom and the top" of the fraction. This indicates a misunderstanding of the identity property for addition and subtraction. The identity for addition and subtraction is 0, not 1, so subtracting <math>\frac{2}{2}</math> in this way would not generate an equivalent fraction. One way to help a student realize that an error has been made would be to directly compare <math>\frac{5}{8}</math> to the result of <math>(5 - 2)/(8 - 2)</math>. This can help to show that the two fractions are not equivalent.</i></li> </ul> <p><i>Video C: How do you think about which equivalent fraction to generate?</i></p> <p><i>Possible point to make:</i></p> <ul style="list-style-type: none"> <li><i>There are infinitely many equivalent fractions that can be made, so it is important to be able to think about the few that would be most useful. The point is to generate denominators that are common. This is a situation where methods like considering the least common multiple of the denominators or the product of the denominators could be used.</i></li> </ul>



Detailed description of activity	Comments & other resources
<p>3. Have participants watch <i>Video D</i> in which two teachers use the common numerator strategy to compare <math>2/5</math> and <math>2/3</math>. As participants watch the clip, they should consider:</p> <ul style="list-style-type: none"> <li>• Why is this method also referred to as “same number of parts, but parts of different sizes?”</li> <li>• What representation is useful to explain teachers’ strategies?</li> </ul> <p>If time allows, have participants discuss the focus questions in small groups.</p>	<p><i>Additional questions to ask:</i></p> <ul style="list-style-type: none"> <li>• <i>Two teachers use the same strategy, “using common numerator”, but narrate differently. How does Natalie’s explanation differ from Keisha’s explanation? What aspects of the working definition of fractions do each teacher use?</i> <ul style="list-style-type: none"> <li>○ <i>Natalie: As numerators are the same (considering the same number of parts), I just compared unit fractions, <math>1/5</math> and <math>1/3</math>. If the whole is divided into five equal parts and if the same whole is divided into three equal parts, a part from three equal parts is larger than a part from five equal parts.</i></li> <li>○ <i>Keisha: As numerators are the same, I have two of unit fraction, and then cancel out two. I knew that <math>1/3</math> is larger than <math>1/5</math>, thus <math>2/3</math> is larger than <math>2/5</math>.</i></li> </ul> </li> <li>• <i>How does Keisha know that “<math>1/3</math> is larger than <math>1/5</math>”?</i></li> </ul>
<p>4. Have participants watch <i>Videos E – F</i> in which the teachers discuss strategies for comparing fractions. For each video clip, participants should name the strategy being used and consider:</p> <ul style="list-style-type: none"> <li>• How does the teacher’s explanation connect with the working definition of a fraction? (slide is available as a Resource in the right side of the online viewer)</li> <li>• When might the strategy be useful?</li> <li>• What representations seem most useful for explaining the strategy?</li> </ul> <div data-bbox="634 841 1018 1128" style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #003366; color: white; padding: 2px;"><b>A working definition of a fraction</b></p> <ul style="list-style-type: none"> <li>- Identify the whole</li> <li>- Make <math>d</math> equal parts</li> <li>- Write <math>1/d</math> to show one of the equal parts</li> <li>- If you have <math>d</math> of <math>1/d</math>, then you have the whole</li> <li>- If you have <math>n</math> of <math>1/d</math>, then you have <math>n/d</math></li> </ul> <ul style="list-style-type: none"> <li>• <math>n</math> and <math>d</math> are whole numbers</li> <li>• <math>d \neq 0</math></li> </ul> <p style="font-size: small; margin-top: 5px;">© 2018 Michigan Department of Education • University of Michigan • 7343-400-4861 • <a href="http://www.mde.state.mi.us">www.mde.state.mi.us</a></p> </div>	<p><i>Video E: More and less than one whole (<math>4/3</math> and <math>14/15</math>)</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: More and less than one whole.</i></li> <li>• <i>Connection to the working definition of a fraction: identify the whole, <math>d</math> equal parts, <math>d</math> of <math>1/d</math>, use the definition of numerator (<math>n</math> of <math>1/d</math>), <math>d</math> of <math>1/d</math> is the whole.</i></li> <li>• <i>When the strategy might be useful: in cases where one fraction is greater than one whole and the other fraction is less than one whole.</i></li> <li>• <i>Representations that seem most useful for explaining: area models and number lines.</i></li> </ul> <p><i>Video F: Distance from one whole (<math>3/4</math> and <math>14/15</math>)</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: Distance from one whole</i></li> <li>• <i>Connection to the working definition of a fraction: If you have <math>d</math> of <math>1/d</math>, then you have the whole.</i></li> <li>• <i>When the strategy might be useful: in situations in which both fractions are one part (or a few parts) away from the whole.</i></li> <li>• <i>Representations that seem most useful for explaining: area models and number lines.</i></li> </ul>

**Part 4: Analyzing students' comparison strategies (~20 minutes)**

<p><b><u>Goals</u></b></p> <ul style="list-style-type: none"> <li>• Participants will be able to identify fraction comparison strategies shown in student work, evaluate whether these strategies are valid, and identify follow-up questions to ask children about their strategies.</li> <li>• Participants will be able to generate follow-up questions to ask children about their fraction comparison strategies.</li> </ul>	<p><b><u>Instructional sequence</u></b></p> <ol style="list-style-type: none"> <li>1. Introduce Part 4.</li> <li>2. Anticipate the strategies and representations that students will use to compare fractions.</li> <li>3. Analyze student work on comparing fractions problems.</li> <li>4. (<i>Optional</i>) Watch the video in which teachers discuss the challenges of using records to learn about student thinking.</li> </ol>	<p><b><u>Resources</u></b></p> <ul style="list-style-type: none"> <li>• Handouts: Student work 1- 6</li> </ul> <p><b><u>Supplements</u></b></p> <ul style="list-style-type: none"> <li>• Video (03:08): Challenges in getting at student thinking</li> </ul>
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Detailed description of activity	Comments & other resources
<p>1. Introduce Part 4: The work participants have done unpacking and categorizing fraction-comparison strategies provides a foundation for anticipating and analyzing student thinking.</p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #1a3d4d; color: white; padding: 2px;">Comparison strategies</p> <ol style="list-style-type: none"> <li>1. More of the same-size parts (common denominator)</li> <li>2. Same number of parts but parts of different sizes (common numerator)</li> <li>3. More and less than one-half or one whole</li> <li>4. Distance from one-half or one whole (could be distance more than or distance less than)</li> </ol> <p style="text-align: right; font-size: small;">(Van de Walle, 2007)</p> </div> <p>Introduce the task: <i>Early in their unit on fractions, a group of fifth-grade students compared the following pairs of fractions:</i></p> <p>A. <math>2/10</math> and <math>3/4</math></p> <p>B. <math>3/7</math> and <math>4/7</math></p> <p>C. <math>5/8</math> and <math>5/9</math></p> <p>With a partner, have participants anticipate the strategies and representations students used to compare each pair of fractions. Participants should use the strategy names on the Comparison Strategies slide. Encourage participants to explain why they believe those strategies and representations are likely to be used.</p>	<p><i>If time is limited, considering doing a light treatment with the anticipation work or perhaps omitting that component altogether. It is important to have adequate time for participants to consider the student work.</i></p>

Detailed description of activity	Comments & other resources
<p>2. Distribute samples of student work on these tasks. For each problem for each student, the pairs should consider:</p> <ul style="list-style-type: none"> <li>• What strategy was used? Does the strategy correspond with our prediction? If the student used a different strategy than predicted, is it a fitting choice?</li> <li>• Did the student use a strategy that is not on the slide? If so, is the strategy mathematically valid? Why or why not?</li> <li>• If you could ask the student a question or pose a follow-up problem, what would it be and why?</li> </ul> <p>As time permits, when partners are finished analyzing the student work, elicit some comments in whole group.</p>	<p><i>Student work 1:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: (A) not explicit; (B) common denominator; (C) common numerator</i></li> <li>• <i>Representation: pattern blocks</i></li> <li>• <i>Follow-up questions: (A) How did you know that <math>2/10</math> is smaller from the drawing? (C) How did you know that <math>5/8</math> is bigger than <math>5/9</math>?</i></li> </ul> <p><i>Student work 2:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: Distance from the whole</i></li> <li>• <i>Representation: Not specific</i></li> <li>• <i>Follow-up questions: Are the take-away parts the same size?</i></li> </ul> <p><i>Student work 3:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: Comparing sizes/areas from pictures (strategy not listed in the slide)</i></li> <li>• <i>Representation: Area models</i></li> <li>• <i>Follow-up questions: Were there any difficulties with drawing a picture?</i></li> </ul> <p><i>Student work 4:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: (A) Distance from the whole; (B) same denominator; (C) same numerator</i></li> <li>• <i>Representation: Not specific</i></li> <li>• <i>Follow-up questions: How did you know that <math>3/4</math> is closer to the whole than <math>2/10</math>?</i></li> </ul> <p><i>Student work 5:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: The number of pieces</i></li> <li>• <i>Representation: Not specific</i></li> <li>• <i>Follow-up questions: Is it always true that the fraction with more pieces is bigger? In the case of C, there are 5 pieces of each fraction. How would you explain why <math>5/8</math> is bigger?</i></li> </ul> <p><i>Student work 6:</i></p> <ul style="list-style-type: none"> <li>• <i>Strategy: Not specific (all answers are correct)</i></li> <li>• <i>Representation: Not specific</i></li> <li>• <i>Follow-up questions: Could you explain your reasoning for each comparison?</i></li> </ul>

Detailed description of activity	Comments & other resources
	<p><i>Analyzing student work on three fraction comparison problems, participants might notice that:</i></p> <ul style="list-style-type: none"> <li>• <i>Some students use different strategies according to problems (student works 1 and 4), while others use the same strategy regardless of problems (student works 2, 3, 5, 6, and 7)</i></li> <li>• <i>Students do not explain their reasoning in detail.</i></li> <li>• <i>Students do not specify the representation for each comparison.</i></li> <li>• <i>Students use area models rather than set models or number lines.</i></li> </ul>
<p>4. (Optional) If time permits and it is useful, have participants watch the <i>video</i>, in which teachers discuss the challenges of learning about the strategies that students use.</p>	<p><i>Challenges of learning about the strategies that students use include:</i></p> <ul style="list-style-type: none"> <li>• <i>Difficulties understanding students' reasoning because of lack of explanations. For example, what is the meaning of "one away" or "two away"?</i></li> <li>• <i>All fraction comparisons do not need to be done with the same strategy.</i></li> <li>• <i>Some students may not be using the greater than or less than symbols correctly.</i></li> <li>• <i>Students have a 50% chance of getting the right answer. It is hard to know whether they actually understand the comparison.</i></li> </ul>

**Part 5: Showing and explaining equivalence (~25 minutes)**

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
<ul style="list-style-type: none"> <li>• Participants will be able to:               <ul style="list-style-type: none"> <li>○ explain what it means for two fractions to be equivalent with respect to particular representations;</li> <li>○ show and explain how to generate equivalent fractions; and</li> <li>○ explain the limitations of representing equivalence using particular representations.</li> </ul> </li> </ul>	<ol style="list-style-type: none"> <li>1. Introduce Part 5, explain the importance of equivalence, and examine equivalent fractions using excerpts from mathematics textbooks.</li> <li>2. Narrate the use of representations for equivalence.</li> </ol>	<ul style="list-style-type: none"> <li>• Handout: Textbook examples of representations of fraction equivalence</li> </ul> <p><u>Supplements</u></p> <ul style="list-style-type: none"> <li>• Math notes: Methods for generating and explaining equivalent fractions</li> </ul>

Detailed description of activity	Comments & other resources
<p>1. Introduce Part 5: Equivalence is a fundamental idea that underlies work in many mathematical areas. Equivalence is central for understanding and working with fractions, as well as for other areas of mathematics, like: algebra (e.g., when solving equations equivalent equations are created), place value (e.g., that 73 is the same as 6 tens and 13 ones), or geometry (e.g., two angles with a difference that is a multiple of 360 degrees are equivalent).</p> <p>Processes for generating equivalent fractions can be challenging for students to understand. One way to help students make sense of numerical procedures is to connect them to a representation.</p> <p>In this part, participants explore representing equivalence and explaining the processes through which equivalent fractions can be generated.</p> <p>Distribute <i>Handout: Textbook examples of representations of fraction equivalence</i>. The handout contains examples of equivalent fractions taken from elementary school textbooks (also shown on the slide in the viewer).</p>	<p><u>CCSSM Link</u>: <i>Recognizing that two fractions are equivalent using a visual model and being able to generate equivalent fractions is a standard that begins in Grade 3 and continues into Grade 4 (3.NF.3, 4.NF.1).</i></p> <p><i>Possible responses to the focus questions for each example:</i></p> <p><i>Example A:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means: The fraction pieces cover the same area.</i></li> <li>• <i>Generating equivalent fractions: Covering a given fraction piece with some number of another-sized fraction piece.</i></li> <li>• <i>Limitations/Challenges: Will only work for particular fractions. For example, if you don't have fraction pieces that are 1/10, then you cannot show that <math>5/10 = 1/2</math>.</i></li> </ul>

Detailed description of activity	Comments & other resources
<p>Have participants work in pairs to examine the textbook examples and discuss the following questions for each:</p> <ul style="list-style-type: none"> <li>• What does it mean for two fractions to be equivalent in that representation?</li> <li>• How can the representation be used to generate equivalent fractions?</li> <li>• What are the limitations or challenges of representing equivalent fractions in this way?</li> </ul> <p>If time, discuss one or two of the representations in whole group.</p> <div data-bbox="793 305 1171 587" data-label="Image"> </div>	<p><i>Example B:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means:</i> The wholes contain the same number of objects and the fractions are represented by the same subset of such objects. In this case, the same number of objects in each set are red.</li> <li>• <i>Generating equivalent fractions:</i> Dividing the whole into subsets containing equal numbers of objects.</li> <li>• <i>Limitations/Challenges:</i> Shows only a limited number of equivalent fractions. For example, if you have a set of 12 circles and 6 are red, you can show that <math>\frac{1}{2}</math>, <math>\frac{2}{4}</math>, <math>\frac{3}{6}</math>, <math>\frac{6}{12}</math> are equivalent fractions. You cannot show using the set of 12 circles that <math>\frac{7}{14}</math> is also equivalent.</li> </ul> <p><i>Example C:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means:</i> The two fractions each represent the same area (given that the wholes are identical in size)</li> <li>• <i>Generating equivalent fractions:</i> One representation can be partitioned to show an equivalent fraction.</li> <li>• <i>Limitations/Challenges:</i> Using grid lines to partition generates a limited number of equivalent fractions.</li> </ul> <p><i>Example D:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means:</i> The same point on the number line (or the same directed distance from zero), given that the distances between 0 and 1 are identical).</li> <li>• <i>Generating equivalent fractions:</i> Intervals can be repeatedly subdivided to show additional equivalent fractions.</li> <li>• <i>Limitations/Challenges:</i> Partitioning the interval into equal size parts may be difficult for students.</li> </ul> <p><i>Example E:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means:</i> The fractions correspond to the same length.</li> <li>• <i>Generating equivalent fractions:</i> Measures can be repeatedly subdivided.</li> <li>• <i>Limitations/Challenges:</i> Rulers are pre-partitioned and therefore only demonstrate that particular fractions are equivalent.</li> </ul>

Detailed description of activity	Comments & other resources
	<p><i>Example F:</i></p> <ul style="list-style-type: none"> <li>• <i>Equivalent means: You can transform one to the other by multiplying (or dividing) the numerator and denominator by the same non-zero number.</i></li> <li>• <i>Generating equivalent fractions: Either multiplying (or dividing) both the numerator and denominator by the same non-zero number.</i></li> <li>• <i>Limitations/Challenges: Does not help students understand WHY the procedure works.</i></li> </ul>
<p>2. One way to help students make sense of numerical procedures is to connect them to a representation. Have partners practice narrating the use of one of the representations for equivalence shown in the handout to explain why the procedure of multiplying the numerator and denominator by the same non-zero number (i.e., the procedure shown in Textbook Example F) works for finding equivalent fractions.</p> <p>If it is useful for your participants, distribute the <i>Math notes</i> document on methods for generating and explaining equivalent fractions.</p>	

**Part 6: Wrap up (~5 minutes)**

<u>Goals</u>	<u>Instructional sequence</u>	<u>Resources</u>
<ul style="list-style-type: none"> <li>Participants will understand ways of connecting the session content to their classroom.</li> </ul>	<ol style="list-style-type: none"> <li>Summarize the session.</li> <li>Explain the Classroom Connection Activities.</li> </ol>	

Detailed description of activity	Comments & other resources
<p>1. Summarize the session: This session provided opportunities to examine different strategies for comparing fractions as a way to:</p> <ul style="list-style-type: none"> <li>unpack fundamental mathematical ideas</li> <li>consider and classify how students compare fractions; and</li> <li>work on teaching practices related to the use of representations.</li> </ul> <p>The session also provided opportunities to develop understanding related to the representation of equivalence and the use of equivalence in a common method of comparing fractions.</p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center;"><b>Summary</b></p> <p>In this session, you:</p> <ul style="list-style-type: none"> <li>Examined strategies for comparing fractions through analysis of math tasks and student work</li> <li>Developed understandings related to representations of equivalence and the use of equivalence in a common method of comparing fractions</li> </ul> <p><small>This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License. <a href="https://creativecommons.org/licenses/by-nc/4.0/">https://creativecommons.org/licenses/by-nc/4.0/</a> © 2018 Mathematics Teaching and Learning to Teach • School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtl@umich.edu</small></p> </div>	
<p>2. Distribute the <i>handout</i> you customized with selected Classroom Connection Activities and accompanying documents described in the following table.</p> <p><u>Required:</u></p> <ul style="list-style-type: none"> <li>Reflection on the use of public recording space in classroom</li> <li>Narrating fraction comparison tasks using area models and set models</li> </ul> <p><u>Optional:</u></p> <ul style="list-style-type: none"> <li>Explaining solutions to a fraction-of-an-area task involving multiple wholes</li> <li>Reading on strategic competence from National Research Council's (2001) <i>Adding it Up</i></li> </ul>	<p><i>While the CCA may seem "academic" compared to others in the module, it is important to signal that the reflective writing is a luxury of sorts, one that teachers too seldom make time for, giving participants the opportunity to look across the records they have collected to notice they ways in which they have changed and improved, as well as space for continued improvement.</i></p> <p><i>Ask participants to upload their reflective paper or to bring you a copy of it so that you can review what participants have been learning and think about the module and your facilitation.</i></p> <p><i>The reading is an additional resource that help participants notice what they have been learning over the course of the professional development sessions.</i></p>