Math Notes: Strategies for Comparing Fractions

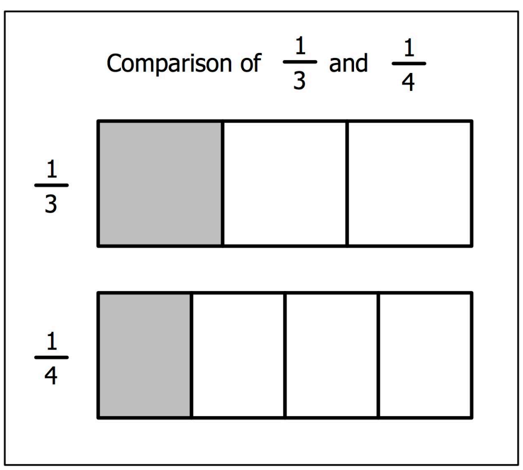
Comparing fractions is a focus in the upper elementary grades. There are four common strategies that can be used to compare fractions (Van de Walle, Karp, & Bay-Williams, 2009) including:

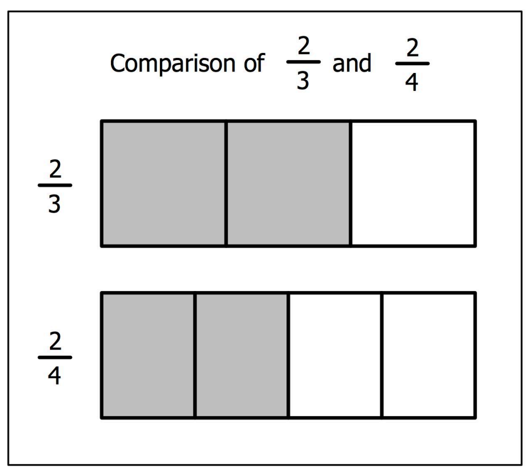
- Same number of parts, but parts of different sizes (common numerator)
- More and less than a benchmark such as one-half or one whole
- Distance from a benchmark such as one-half or one whole (could be distances more than or less than)
- More of the same-sized part (common denominator).

Importantly, these strategies do not rely upon a particular representation such as an area model, number line, or set model. This document discusses these four strategies with a focus on the common denominator strategy. A key point to keep in mind is that comparison of fractions is meaningful only when the fractions refer to the same-sized whole.

Strategy #1: Same number of parts, but parts of different sizes

When two fractions have the same numerator, the greater fraction is the one with the lesser denominator. For example, as illustrated below, if one whole is divided into three equal parts and a second same-sized whole is divided into four equal parts, the size of each of the three equal parts is greater than the size of each of the four equal parts. In other words, the fewer the number of equal parts into which the whole is divided, the greater the size of the part. Thus, it can be seen that $\frac{1}{3}$ is greater than $\frac{1}{4}$ because thirds are greater than fourths. The argument is similar for comparing $\frac{2}{3}$ and $\frac{2}{4}$.





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This is a strategy that works for comparing any two fractions. While not commonly taught, it is possible to find a common numerator for two fractions by multiplying the numerator and denominator of each fraction by a strategically chosen non-zero number so that each fraction is rewritten as an equivalent fraction with the common numerator. For example, to compare $\frac{2}{5}$ and $\frac{3}{4}$, each fraction is rewritten as an equivalent fraction with a numerator of 6 and then the denominators are compared:

$$\frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$$

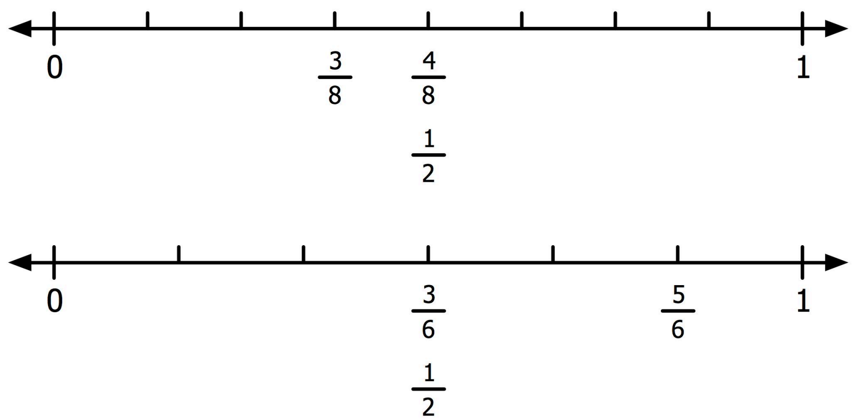
$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$$

Because $15 > 8$, this means that $\frac{6}{15} < \frac{6}{8}$, which means that $\frac{2}{5} < \frac{3}{4}$

Children, however, often develop only partial understandings of this strategy and will compare fractions based on their denominators without finding a common numerator. For example, in a 2010 study of fourth and sixth grade students' strategies for comparing fractions, Julie McNamara and Meghan Shaughnessy found that 40% of fourth graders and 34% of sixth graders argued that $\frac{5}{6}$ is greater than $\frac{7}{8}$. Further, a large proportion of those students that indicated that $\frac{5}{6}$ is the greater fraction reasoned that sixths are greater than eighths and so $\frac{5}{6}$ must be greater than $\frac{7}{8}$. While these students were correctly reasoning about the meaning of the denominator, they were applying the strategy in a case where the numerators were not the same.

Strategy #2: More and less than a benchmark such as one-half or one whole

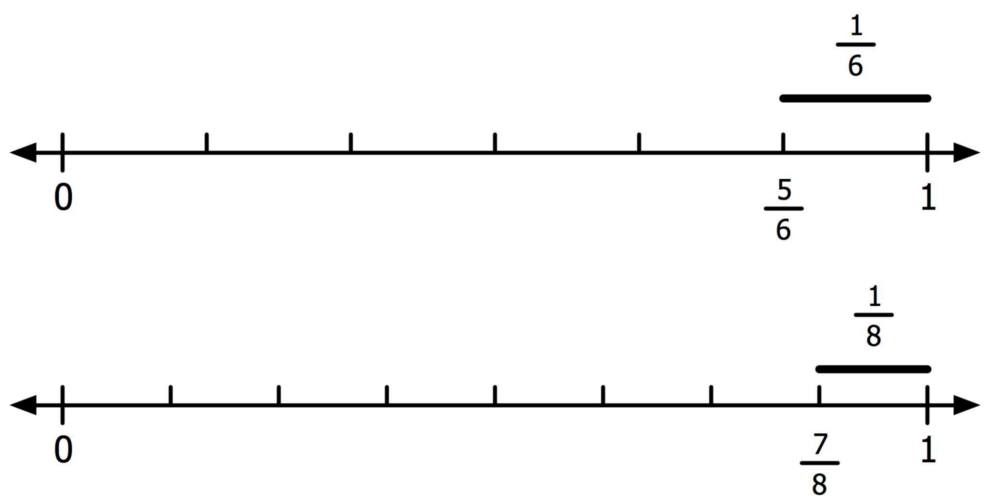
Comparing fractions to a benchmark such as one-half or one whole is another strategy for comparing two fractions. For example, when comparing $\frac{3}{8}$ and $\frac{5}{6}$, we know that $\frac{3}{8}$ is less than $\frac{1}{2}$ and that $\frac{5}{6}$ is more than $\frac{1}{2}$. Thus, $\frac{5}{6}$ is the greater fraction. Geometrically, we can illustrate this strategy by placing both fractions on the number line as shown below.



While any two fractions that represent distinct numbers can be compared to a benchmark, this is a strategy that works particularly well when one fraction is more than a common benchmark such as $\frac{1}{2}$ or 1 and the other fraction is less than this benchmark. Given two fractions greater than $\frac{1}{2}$ but less than 1, finding an appropriate benchmark can be problematic. When comparing $\frac{5}{6}$ and $\frac{13}{16}$, it is difficult to determine a benchmark fraction.

Strategy #3: Distance from a benchmark such as one-half or one-whole

The second strategy, comparing two fractions to a benchmark, is limited when both fractions are either greater than or less than the chosen benchmark. A strategy in these cases is to compare the distances of the fractions from the benchmark. For example, $\frac{5}{6}$ and $\frac{7}{8}$ are both less than the benchmark 1 and so we can compare these two fractions by considering their distance from 1. $\frac{5}{6}$ is $\frac{1}{6}$ less than 1 and $\frac{7}{8}$ is $\frac{1}{8}$ less than 1. We can then compare $\frac{1}{6}$ and $\frac{1}{8}$ using our strategy for comparing two fractions that have the same number of parts, but parts of different sizes. Since $\frac{1}{6}$ is greater than $\frac{1}{8}$, then $\frac{1}{6}$ is a greater distance from 1 than $\frac{1}{8}$. Therefore, $\frac{7}{8}$ is greater than $\frac{5}{6}$ (or $\frac{5}{6}$ is less than $\frac{7}{8}$). Geometrically, we can illustrate this strategy using the number line as shown below.



Strategy #4: More of the same-sized part (common denominator)

It is sometimes useful to find a common denominator to compare fractions. Once fractions have the same denominator, the greater fraction is the one with the greater numerator. Numerically, a typical procedure for comparing fractions by finding a common denominator involves multiplying the numerator and denominator of each fraction by a strategically chosen non-zero number so that each fraction is rewritten as an equivalent fraction with the common denominator. This need not be the least common denominator. For example, to compare $\frac{2}{5}$ and $\frac{3}{8}$, each fraction is rewritten as an equivalent fraction with a denominator of 40 and then the numerators compared:

$$\frac{2}{5}=\frac{2\times 8}{5\times 8}=\frac{16}{40}$$

$$\frac{3}{8}=\frac{3\times 5}{8\times 5}=\frac{15}{40}$$

Because 16 > 15, this means that $\frac{16}{40}>\frac{15}{40}$,

which means that $\frac{2}{5}>\frac{3}{8}$.

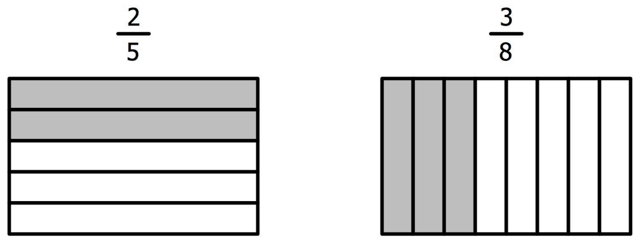
This method is useful for a number of reasons. It works in general and, except for the potential difficulty of the multiplication, is fairly easy to execute. Part of the simplicity of the method is due to the fact that a common denominator can be always be found by multiplying the denominators of the fractions you wish to compare. Furthermore, common denominators are also used when adding and subtracting fractions, which makes this method useful beyond comparison.

However, there are often problems with the ways in which this method is taught and used in school. For example, a common denominator strategy might be automatically used when other strategies would be more efficient or provide opportunities to develop number sense. Another issue is that the numeric procedure is often taught without support for understanding how or why the procedure works resulting in students misapplying the numeric procedure.

An area model can be used when explaining the procedure for comparing fractions by finding a common denominator. The steps below use the above example of comparing $\frac{2}{5}$ and $\frac{3}{8}$ to illustrate one way an area model can be used to explain this procedure.

1. Represent each fraction.

The first step is to represent each fraction. When comparing fractions, it is important that each fraction be represented using the same-sized whole.

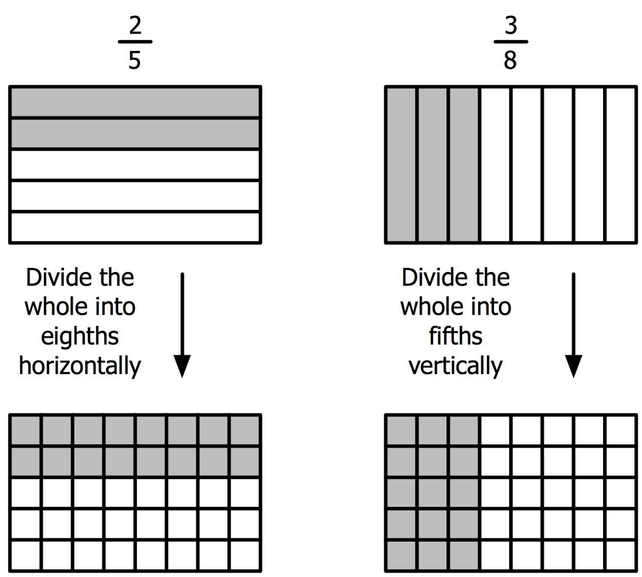


Two things are worth noting about the above diagrams. First, it is not obvious which has the greater shaded area: $\frac{3}{8}$ has more parts than $\frac{2}{5}$, but each part is smaller in size. Second, the partitions in each rectangle have been made in different directions. That is, for $\frac{2}{5}$, the partitions have been made horizontally; and for $\frac{3}{8}$, vertically. When using an area model to explain how to find a common denominator, it is not necessary to make one fraction's partitions horizontally and the other's vertically. However, as will be seen in the next step, doing so creates a convenient method for partitioning each rectangle into parts of the same size (i.e., finding a common denominator).

2. Partition each whole into the same number of equal-sized parts.

As mentioned above, because the wholes have not been divided into the same number of equal parts, the parts in each whole are of different sizes. This means that simply counting and comparing the number of shaded parts in each rectangle does not determine the greater fraction. However, if the wholes are further divided so that they do have the same number of equal-sized parts (i.e., a common denominator), then the shaded areas can be easily compared by counting the number of parts in each.

Because the direction of the partitions in each diagram are perpendicular to each other, the rectangles can easily be divided into the same number of equal-sized parts by "copying" one rectangle's partitions onto the other. In other words, by making eight (equally spaced) vertical partitions on the two-fifths diagram, and five (equally spaced) horizontal partitions on the three-eighths diagram, as shown below:



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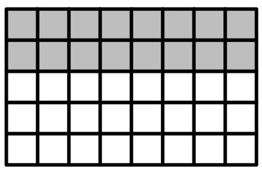
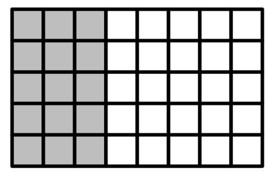
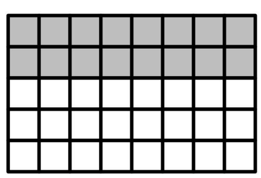
Although the area of each shaded region remains unchanged, each rectangle has now been divided into a greater number of equal-sized parts. Each of the 5 parts in the $\frac{2}{5}$ diagram was divided into 8 equal parts, and each of the 8 parts in the $\frac{3}{8}$ diagram was divided into 5 equal parts. The next step is to determine the equivalent fraction represented by each shaded region.

3. Determine the equivalent fraction that corresponds to each of the shaded regions.

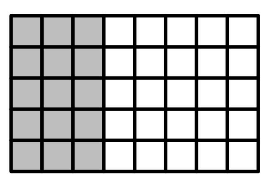
Equivalent fraction for $\frac{2}{5}$:	Equivalent fraction for $\frac{3}{8}$:
	
There are 40 equal-sized parts in the whole, 16 of which are shaded. This diagram can be mapped onto the numeric procedure for generating equivalent fractions (i.e., $\frac{2}{5} = \frac{2\text{x}8}{5\text{x}8} = \frac{16}{40}$): The additional partitioning created 8 times as many parts in the whole (5 x 8), because each of the 5 original parts was divided into 8 equal parts. This partitioning resulted in 8 times as many shaded parts (2 x 8), because each of the 2 shaded parts was divided into 8 equal parts. Because $\frac{2}{5}$ and $\frac{16}{40}$ correspond to the same area, this shows that $\frac{2}{5}$ is equivalent to $\frac{16}{40}$.	Similarly, in this diagram, there are now 40 equal-sized parts in the whole, 15 of which are shaded. Each of the original 8 parts in the whole was divided into 5 equal parts (which means that each of the 3 shaded parts was divided into 5 equal parts). This corresponds to multiplying the numerator and denominator each by 5 (i.e., $\frac{3}{8} = \frac{3\text{x}5}{8\text{x}5} = \frac{15}{40}$). Because $\frac{3}{8}$ and $\frac{15}{40}$ correspond to the same area, this shows that $\frac{3}{8}$ is equivalent to $\frac{15}{40}$.

4. Compare the number of shaded parts.

Because they have the same-sized equal parts (i.e., the same denominators), $\frac{16}{40}$ and $\frac{15}{40}$ can be compared by comparing the number of shaded parts in each diagram (i.e., comparing the numerators).



$$\frac{2}{5} = \frac{16}{40}$$



$$\frac{3}{8} = \frac{15}{40}$$

Because 16 is greater than 15, $\frac{16}{40}$ is greater than $\frac{15}{40}$, which means that $\frac{2}{5}$ is greater than $\frac{3}{8}$.

References

McNamara, J., & Shaughnessy, M. M. (2010). *Beyond pizzas & pies: 10 essential strategies for supporting fraction sense, grades 3-5*. Sausalito, CA: Math Solutions.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2009). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Columbus, OH: Allyn & Bacon.