

## Geometric Measurement and Spatial Reasoning in Elementary Mathematics Teaching

**Building  
Blocks**



**DTE@**  
**MATHEMATICS**



## Goals for all DTE modules

Integrated attention to four core elements of elementary mathematics teaching:

1. Mathematics geared to teaching
2. Student thinking about mathematics
3. High-leverage mathematics teaching practices
4. Approaches for learning from and improving teaching

**Goals for this module:  
Geometric Measurement and Spatial Reasoning in  
Elementary Mathematics Teaching**

- Mathematics: understanding measurement, spanning grades pre-K to 6
- Student thinking: trajectories for noticing students' development of measurement knowledge and skills
- Teaching: enhancing skills with formative assessment and task/curriculum analysis
- Learning from practice: studying teaching and learning of geometric measurement through anecdotal notes and video

## The ten sessions

1-3: Introduction and length

4-6: Area

7-9: Volume

10: Bringing it all together

## Classroom Connection Activities: Overview

- “Professional homework” designed to:
  - Connect professional development content with classroom teaching
  - Extend thinking about the content of the present and previous sessions

## Session 1: Length Learning Trajectory - Mathematical goals

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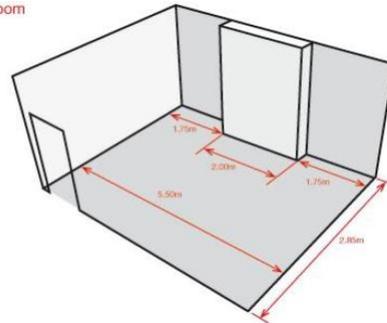
## Overview of Session 1

- Working on a length measurement problem
- Analyzing length measurement in standards for student learning
- Unpacking the mathematics of length measurement
- Introducing Classroom Connection Activities

## How long is the room? (Estimating)

- Really...how long? How wide?
  - Estimate. Write it in your notebooks
  - Think: How did you estimate?
  - What did you have to know and be able to do?
  - Write a brief summary as notes
  - Brief sharing

Example Room



## How long is the room? (Measuring)

- Structure: Individual, pairs, whole group
- Choose a personal ruler\* to measure the length of the room

\*any object you have with you, including...  
any part of your body!

Then, with partner, measure your personal rulers using  
standard ruler and compute the length of the room in  
standard units

- Go measure!

## Our measurements of the room length

- What did you use for a personal ruler?
- What was the length of the room using your personal ruler?
- What was the length of the room using standard units?

## Making sense of measures of length

- Why did we get different answers?
  - How did different personal ruler selections and methods affect the results?
  - What differences are or are not acceptable?
- How did you deal with partial units?
- Errors—did they propagate multiplicatively?
- How did the results compare with your initial estimate? Why? How did you estimate?

## Reflection

- What are the implications for students' measurement activity?
- What did you notice about your and others' use of language, tools, representations, and structure to justify or critique solutions?
- How about mathematical practices?

## Length in the Common Core Standards

<p><b>Measurement and Data: K.MD</b></p> <p><b>Describe and compare measurable attributes.</b></p> <ol style="list-style-type: none"> <li>Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.</li> <li>Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.</li> </ol>	<p><b>Measurement and Data: 1.MD</b></p> <p><b>Measure lengths indirectly and by iterating length units.</b></p> <ol style="list-style-type: none"> <li>Order three objects by length; compare the lengths of two objects indirectly by using a third object.</li> <li>Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.</li> </ol>	<p><b>Measurement and Data: 2.MD</b></p> <p><b>Measure and estimate lengths in standard units.</b></p> <ol style="list-style-type: none"> <li>Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.</li> <li>Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.</li> <li>Estimate lengths using units of inches, feet, centimeters, and meters.</li> <li>Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit. Relate addition and subtraction to length.</li> </ol>
<p><b>Measurement and Data: 2.MD</b></p> <p><b>Relate addition and subtraction to length.</b></p> <ol style="list-style-type: none"> <li>Use addition and subtraction within 100 to solve word problems involving lengths given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number.</li> <li>Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.</li> </ol> <p><b>Represent and interpret data.</b></p> <ol style="list-style-type: none"> <li>Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object.</li> </ol>	<p><b>Measurement and Data: 3.MD</b></p> <p><b>Represent and interpret data.</b></p> <ol style="list-style-type: none"> <li>Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch.</li> </ol> <p><b>Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.</b></p> <ol style="list-style-type: none"> <li>Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.</li> </ol>	<p><b>Measurement and Data: 4.MD</b></p> <p><b>Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.</b></p> <ol style="list-style-type: none"> <li>Know relative sizes of measurement units within one system of units including km, m, cm. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit.</li> <li>Solve word problems involving distances, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.</li> <li>Apply the area and perimeter formulas for rectangles in real world and mathematical problems.</li> </ol>

## Common Core State Standards for Mathematical Practice: Connected to measurement

2. Reason abstractly and quantitatively
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure



## Common Core State Standards for Mathematical Practice: Measurement at your grades

- Explore CCSS' content standards for length measurement: Your grade and at least the grades before and after
- Write in your notebooks, especially:
  - questions
  - concerns
  - missing pieces



## Measurement stories

...and what they mean for standards and instruction.

## Wally and the rug (Part 1)

The class was about to act out "Jack and the Beanstalk" when Wally and Eddie had a disagreement about two rectangular rugs (Paley, 1981).

**Wally:** The big rug is the giant's castle. The small one is Jack's house.

**Eddie:** Both rugs are the same.

**Wally:** They can't be the same.

**Eddie:** You have to measure it. You need a ruler.

**Wally:** We have a ruler.

**Eddie:** Not that one. Not the short kind. You have to use the long kind that gets curled up in a box.

## Wally and the rug (Part 2)

**Wally:** Use people. People's bodies. Lying down in a row.

**Eddie:** That's a great idea. I never even thought of that.

Wally announced a try-out for "rug measurers."

4 by 3 students. Everyone satisfied.

Next day, Eddie measured again. Himself, Wally, and 2 others, but it came up too short.

**Wally:** You're too short. I mean someone is too short. We need Warren.  
Where's Warren?

Wally was told that Warren was absent.

## Wally and the rug (Part 3)

**Eddie:** Then we can't measure the rug.

**Teacher:** You can only measure the rug when Warren is here?

**Jill:** Because he's longer.

**Deana:** Turn everyone around. Then it will fit.

Eddie rearranged the students. Surprise...unchanged.

**Eddie:** No, it won't work. We have to wait for Warren.

**Deana:** Let me have a turn. I can do it.

**Jill:** You're too big, Deana. Look at your feet sticking out...

## Wally and the rug (Part 4)

**Teacher:** Is there a way to measure the rug so we don't have to worry about people's sizes?

**Kenny:** Use short people.

**Teacher:** And if the short people aren't in school?

**Rose:** Use big people.

**Deana:** Use rulers. Get all the rulers in the room. I'll get the box of rulers .... This isn't enough rulers.

**Wally:** Use the dolls.

**Teacher:** So this rug is ten rulers and two dolls long? (Silence.) Here's something we can do. We can use one of the rulers over again, this way.

## Wally and the rug (Part 5)

**Eddie:** Now you made another empty space.

**Teacher:** Eddie, you mentioned a tape measure before...here.

We stretch the tape along the edge of the rug, and I show the students that the rug is 156 inches long. The lesson is done. The next day Warren is back in school.

**Wally:** Here's Warren. Now we can really measure the rug.

**Teacher:** Didn't we really measure the rug with the ruler?

**Wally:** Well, rulers aren't really real, are they?

## Reflecting on Wally and the rug

What does this tell you about students' thinking  
about measurement?

## Doug, friends, and the room (Part 1)

Third graders were making a map of their classroom (Clements, 1999).

They wished to begin by measuring the room. Pleased, I passed out meter sticks. They began laying these down but soon stopped, puzzled.

“We need more.”

## Doug, friends, and the room (Part 2)

"More meter sticks?" I inquired.

"Yeah. There's not enough."

"Maybe you could work together and solve that."

"No. Even all of 'em wouldn't reach."

"I mean is there a way you could measure with just the meter sticks you have?"

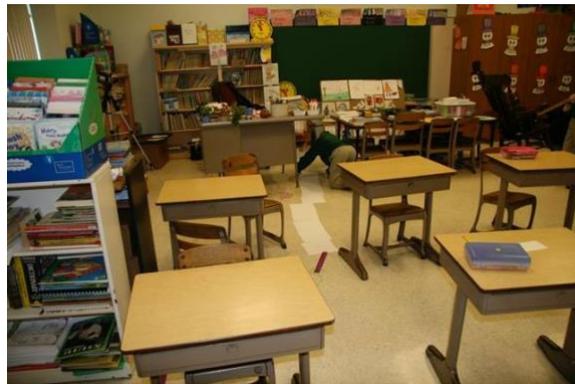
After minutes of futile attempts and useless hints, I believed I was miscommunicating. So I demonstrated.

"How about this? Can you lay a meter stick down, mark the end with your finger, and then move it?"

"Wow! Good idea!"

## Doug, friends, and the room (Part 3)

- Reasoning plays a role
- But, as with number, students find ways to measure, if they have experiences!



## Reflecting on students and the mathematics of measuring

- Length measurement concepts and skills in the Common Core and stories
- What are the underlying math constructs?
- What are the cognitive abilities and understandings students would need to develop to achieve these goals?
  - Reflect on your experiences (today and before)
  - Reflect on your students' experiences
  - Reflect on the measurement stories
- Whole group sharing

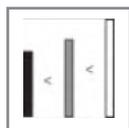
## Length concepts (Part 1)

### Understanding of the attribute of length

Conservation



Transitivity



Equal partitioning

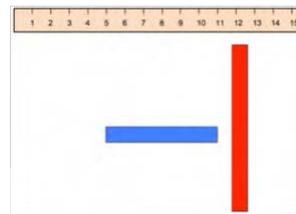


## Length concepts (Part 2)

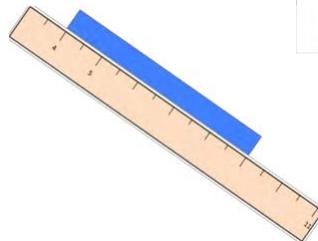
Units and unit iteration



Accumulation of distance  
and additivity



Origin



Relation between (discrete)  
number and measurement

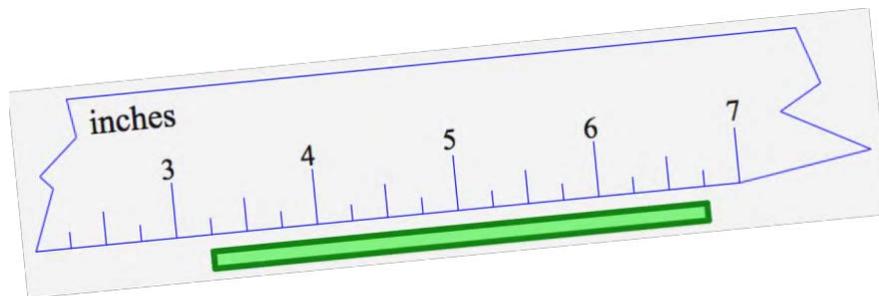


## The Broken Ruler Task (Part 1)

- A “Classic” assessment that can reveal thinking across a wide variety of grade levels
- Watch the video, to:
  - Learn how to administer this assessment, as you will administer it to one or more students (as a part of the Classroom Connection Activities (CCA) you will do before our next session); and
  - Begin a discussion of students' thinking

## The Broken Ruler Task (Part 2)

The task is to tell the length of an object measured with a “broken ruler” (one with no origin, or zero point)



## Why the Broken Ruler?

- Measurement is too often taught as skills, not concepts
- Measurement is often therefore misunderstood by students and teachers
- So, measurement activities often are so well defined and simple, they “mask” misunderstandings that persist for some students for years

## Note taking as you watch videos

Write ideas, words, or phrases about the mathematics  
and the student's thinking as you watch the video

## Discussion of notes taken on the video

- What did you write in your notes?
- How did you decide what to write?
- How did you manage to capture the ideas quickly (shorthand, pictures, etc.)

## The mathematics of length measurement (Part 1)

- Length is a characteristic of an object and can be found by quantifying how far it is between the endpoints of the object
- Distance refers to the empty space between two points

## The mathematics of length measurement (Part 2)

- Measuring consists of two aspects:
  - Identifying a unit of measure and subdividing (mentally and physically) the object by that unit
  - Iterating that unit end to end to measure alongside the object

## Summary

In this session you:

- Determined, compared and applied different measures of length
- Analyzed the ways that length appears in standards for students learning
- Unpacked the mathematics involved in measuring length
- Analyzed students' approaches to task (the Broken Ruler) that you will be trying with your students