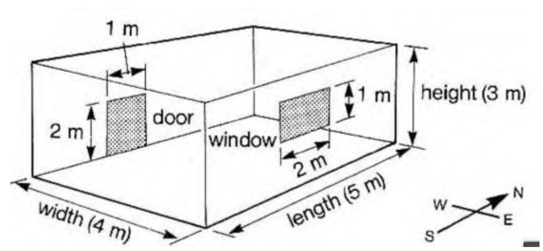
Session 8: Volume Learning Trajectory – Developmental progression

Building Blocks



DTE@ MATHEMATICS

Overview of Session 8

- Discussing what you learned about students' thinking from the CCA assessment tasks
- Unpacking the Developmental Progression of the Learning Trajectory for volume by watching students measure
- Classroom Connection Activity

CCA – Focal tasks from last time

- Volume and Spatial Structuring (3D arrays)
- Piagetian conservation tasks (optional)

CCAs – What did you find?

- In groups of 2-4, discuss your students' responses to the tasks. Think about:
  - What mathematics do they know?
  - How do they think about the math?
  - What differences did you notice?
- Share within your small groups

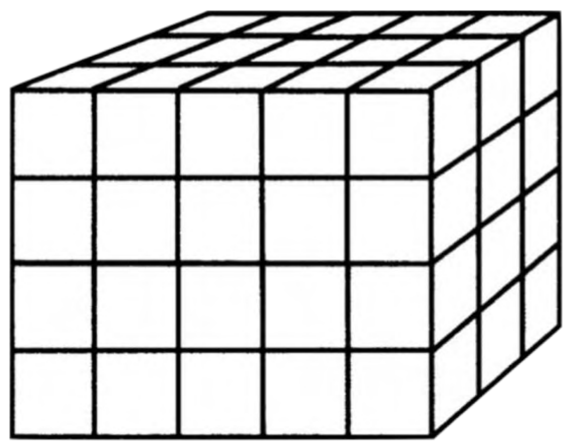
Students' thinking about volume and spatial structuring

What started our investigations?

Spatial structuring: Student responses (Part 1)

"How many cubes to build this?"

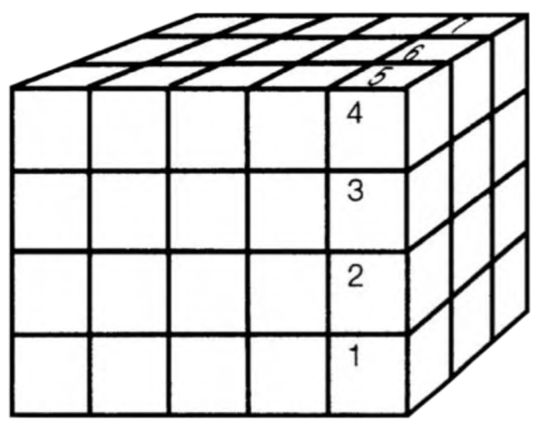
- Counted faces (not on bottom)
- "79" (because double-counted cubes on edges)



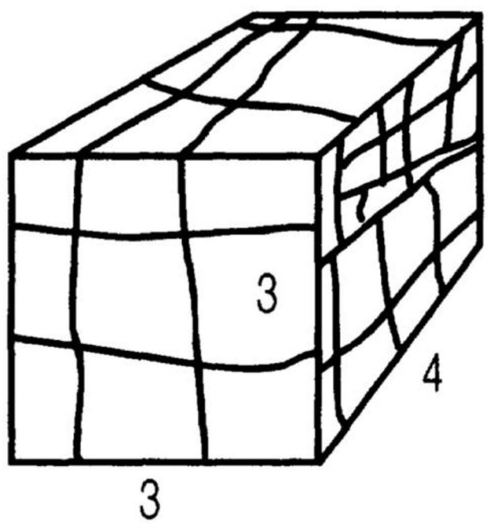
Spatial structuring: Student responses (Part 2)

"How many cubes to build this?"

- The student built bottom layer, got 15
- Then counted the "height" and multiplied 7 x 15 to determine an answer of 105 cubes



Spatial structuring: Student responses (Part 3)



Developmental progression for volume measurement

Schemes

- Filling
- Packing
- Building
- Comparing

Developmental progression

- Volume Quantity Recognizer (VQR)
- Volume Filler (VF)
- Volume Quantifier (VQ)
- Volume Unit Relater and Repeater (VURR)
- Initial Composite 3-D Structurer (VICS)
- 3-D Row and Column Structurer (VRCS)
- 3-D Array Structurer (3D AS)

Volume Quantity Recognizer (VQR)

Filling Scheme

Identifies capacity as attribute
I can pour lots of sand into this can.

General language without comparison
This jar is big.

Packing Scheme

Identifies volume as attribute
This box doesn't hold many toys.

General language without comparison
This box is big.

Building Scheme

Builds with blocks: *I made a big house.*

May not recreate a given shape in size or dimensions

Counts on only one face of cube building

Comparing Scheme

May compare containers, but makes no reference to dimensions:
This glass is big. This one is tiny.

Eventually may compare 1 dimension:
This one's bigger because it's taller.

Volume Filler (VF)

Filling Scheme

Fills container & counts, but may not recognize need for equal size units

Smaller container, fewer scoops - no quantification

Attends to space *filled*, not capacity

Packing Scheme

Fills box with cubes, but leaves gaps. Sometimes only one layer

Eventually fills, but doesn't quantify or use equal-size units

May not recognize "half full"

Building Scheme

May recreate, attending to 1-2 dimensions, but not pattern / plan

Counts multiple faces of cube building without pattern

Comparing Scheme

Compares by aligning 1-2 dimensions
This one holds more, it's longer and wider.

Compares counts, but without accurate recognition of unit size or number
This is big; that is small. Two scoops for this one; one scoop for that one.

Volume Quantifier (VQ)

Filling Scheme

Estimates number of scoops, but may not explicitly maintain unit size

Partitions space (capacity); can recognize "half full"

Packing Scheme

Limited spatial structuring: *counts single units*

Does not recognize need for equal-size units

Recognizes "half full," but may not visualize or calculate total

Building Scheme

Partial understanding of cubes as filling space: *Initially may double-count cubes at corners and ignore internal cubes*

Piaget's "coordination" (integration) of dimensions.

Comparing Scheme

Compares, recognizes 3 dimensions

Directly compares capacity

Attempts to compare count of cubes

Volume Unit Relater & Repeater (VURR)

Filling Scheme

Accurately counts number of scoops

Relates size and number of units explicitly

Packing Scheme

Accurate packing and counting

Relates size and number of units

Able to iterate a unit throughout volume; although may ignore internal cubes

Building Scheme

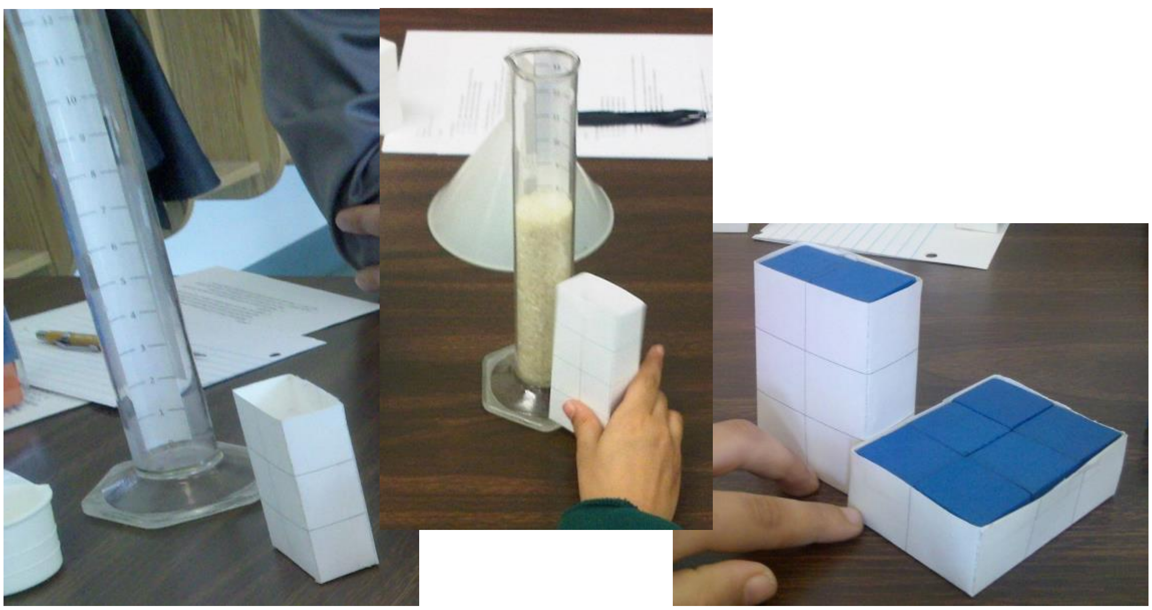
Begins to understand cubes as filling space

Counts cubes, not faces

Comparing Scheme

Begins to relate number and size of units to volume

Relating filling and packing



Initial Composite 3-D Structurer (VICS)

Filling Scheme

Relates number of cubes to cubic units as measured by capacity

Sand filled to the 10 in graduated cylinder would fill a box that holds 10, inch cubes

Packing Scheme

"Sees" rows and columns (but not layers)

Fills/iterates unit to fill space (including internal)

Partitions space; uses units or subunits; visualizes remaining rows or columns

Building Scheme

Understands cubes as filling a space, moves to more sophisticated strategies and additive reasoning

Counts number of cubes in one row/column of 3-D structure, skip counts to get total

Comparing Scheme

Explicitly relates number and size of units to volume

Recognizes that buildings of different shapes but made from same number of cubes could be packed into the same size box

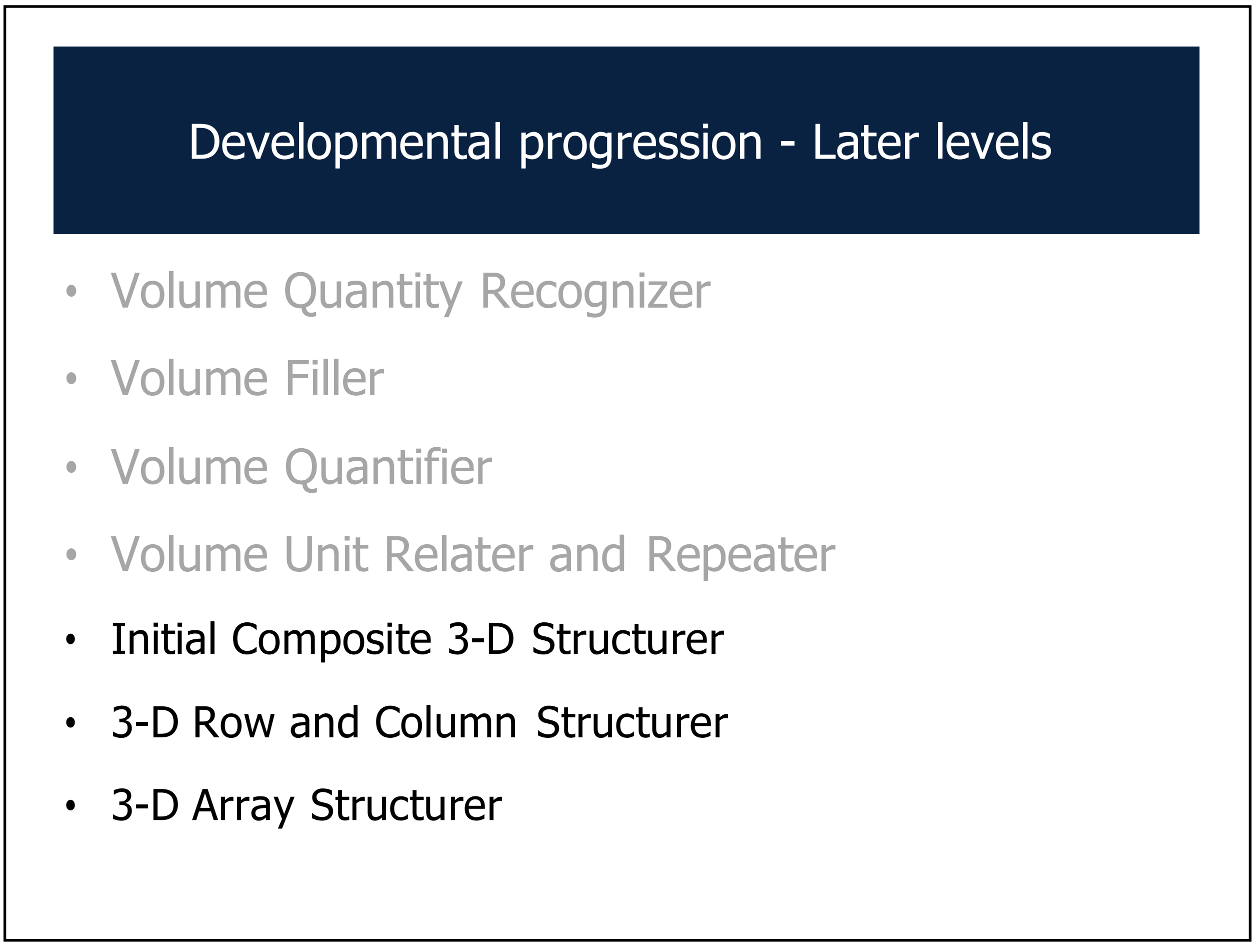
3-D Row and Column Structurer (VRCS)

- Coordinates filling, packing, building schemes of volume
- Additive comparisons (e.g., "this one has 12 more")
- Counts/computes the number of cubes in one layer, and then uses addition or skip counting by layers to determine the total volume
- Operates flexibly on units (cubes), units of units (rows/columns), and units of units of units (layers)

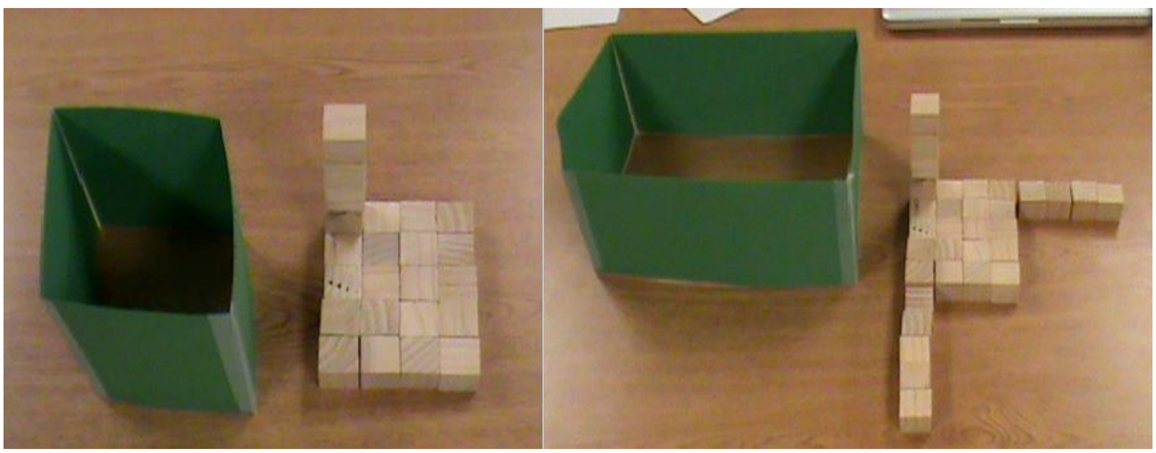
  With perceptual support, can decompose 3-D arrays into other, complex 3-D arrays (not only layers, rows, or columns) and calculate the number of these smaller arrays in the larger array

3-D Array Structurer (3D AS)

- Abstract understanding of the rectangular prism volume formula; makes multiplicative comparisons
- With linear measures or other similar indications of the three dimensions, multiplicatively iterates cubes in a row, column, and/or layer to determine volume
- Visualizes and operates on both horizontal and vertical layers
- Decomposes 3-D arrays into other, complex 3-D arrays (not only layers, rows, or columns) and calculates the number of these smaller arrays in the larger array


Developmental progression - Later levels
Volume Quantity Recognizer
Volume Filler
Volume Quantifier
Volume Unit Relater and Repeater
Initial Composite 3-D Structurer
3-D Row and Column Structurer
3-D Array Structurer

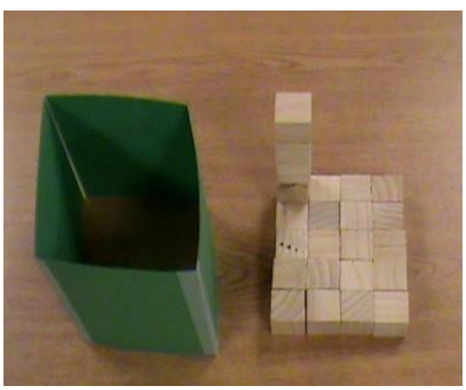
Task 1: Outlined containers

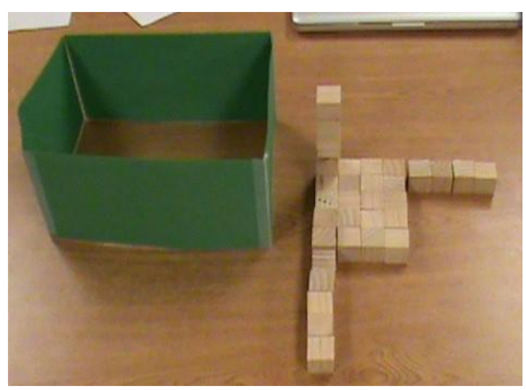


How many blocks would be needed to fill this container?

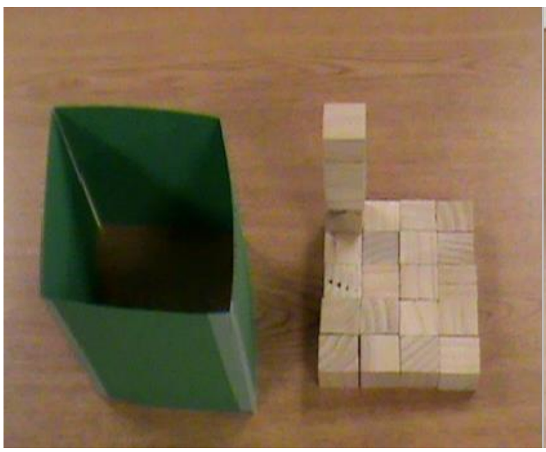
Predict

- How might a Volume Quantifier respond to this task?
- How would a 3-D Row and Column Structurer respond to this task?

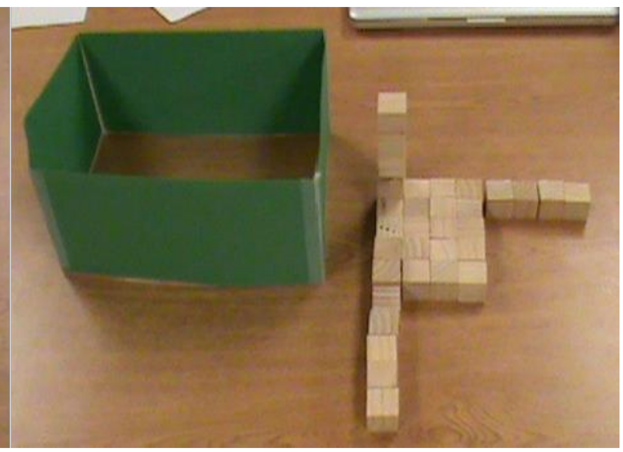




Task 1: Outlined containers
Our results (Grade 4)



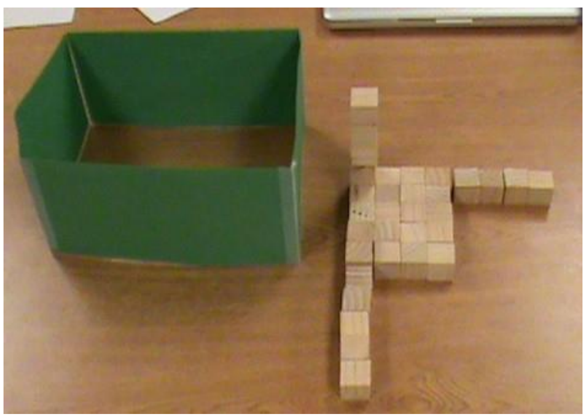
All of the students answered correctly



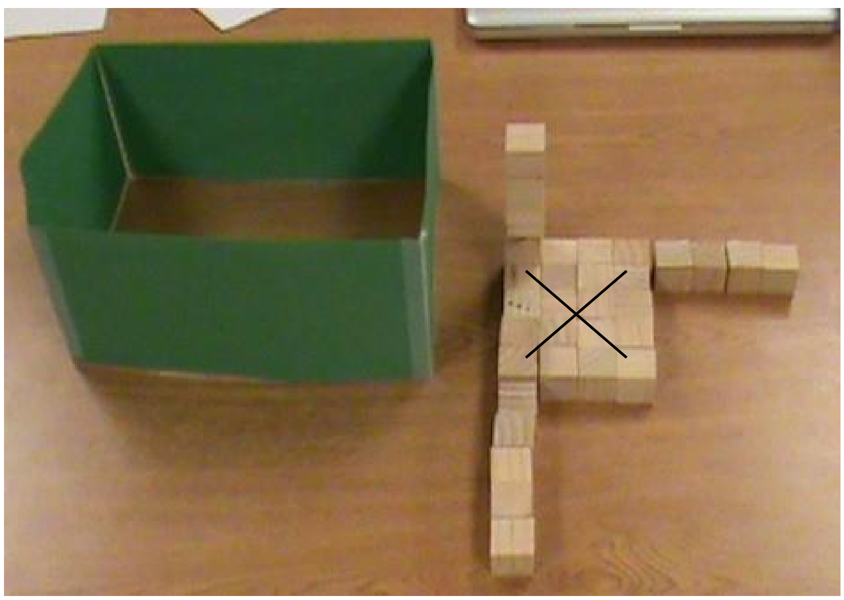
All but one student answered incorrectly

Task 1: Modifying the outlined containers task

- How could you modify this task to make it accessible to these students?



Task 1: One modification to the outlined containers task



Task 1: Connecting the outlined containers task to standards

- Units: 3-D cubes, missing layers/rows/columns
- Task representation: 3D

Connections to CCSSM

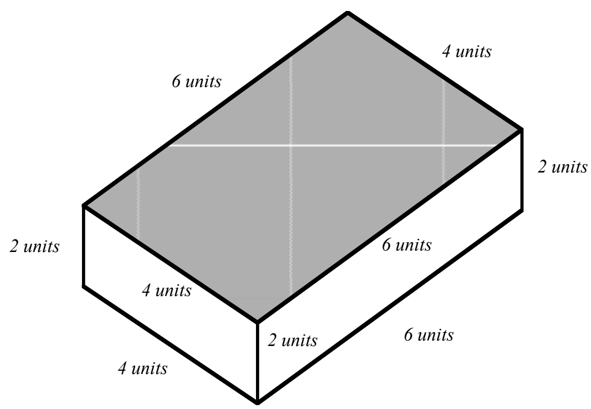
Grade 5

Measure volumes by

- Finding the total number of same-size units of volume required to fill the space without gaps or overlaps
- Viewing 3D shapes as decomposed into layers of arrays of cubes

Relate volume to multiplication and addition and solve real world and mathematical problems

Task 2: Volume of the solid

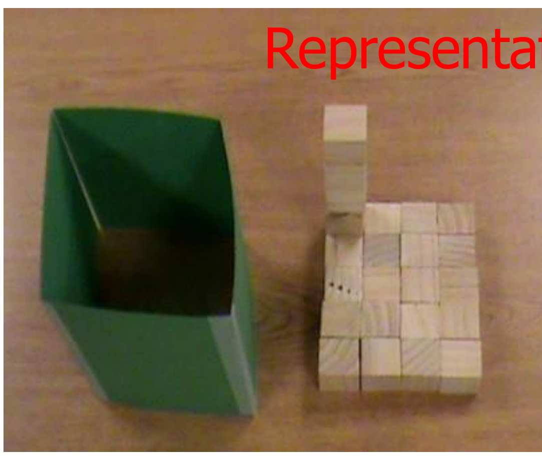


How might a 4th grader respond?

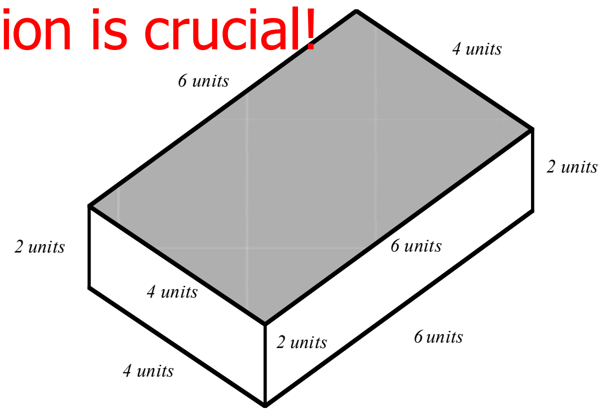
One added all the number labels. The task representation influenced his strategy and squelched his spatial structuring

Outlined containers vs. volume of the solid

Representation is crucial!



All of the students answered correctly



Half of the students added all the number labels

Task 2: Connecting the volume of the solid task with standards

- Units: Numerals as units for linear dimensions
- Task representation: 2D

Connections to CCSSM

Grade 5

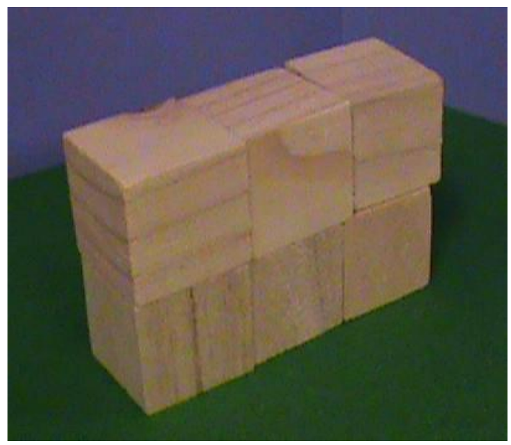
Measure volumes by

- Selecting appropriate units, strategies, and tools
- Counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units
- Viewing 3D shapes as decomposed into layers of arrays of cubes

Relate volume to multiplication and addition and solve real world and mathematical problems

Task 3: Drawing vs. building

- Draw something that has three times the volume
- Build something that has three times the volume

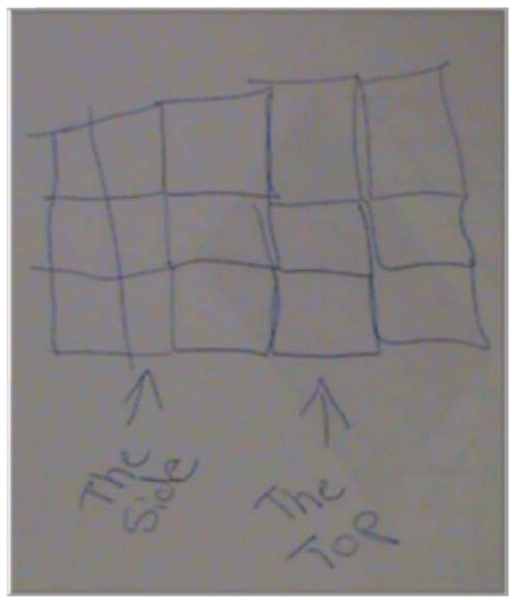


How would an Initial Composite 3-D Structurer respond to this task?

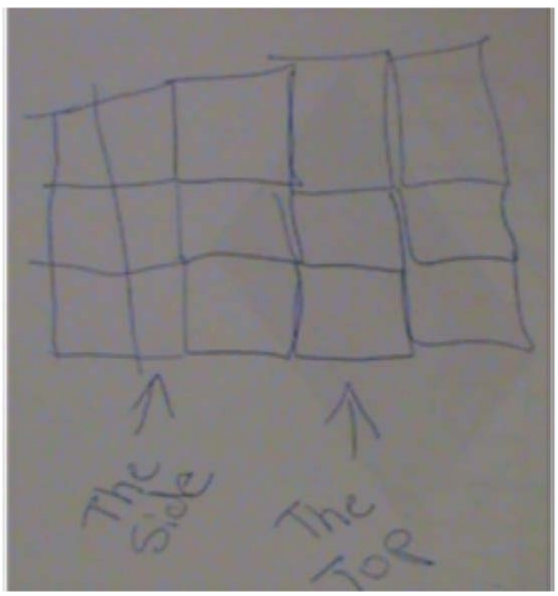
Task 3: Drawing vs. building
Our results at grade 4 (Part 1)

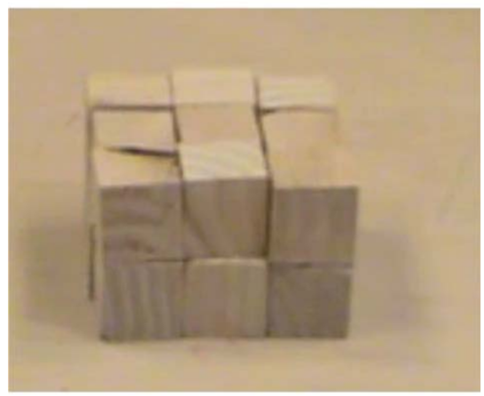
Draw something that has three times the volume

Is he correct? →



Task 3: Drawing vs. building
Our results at grade 4 (Part 2)





Producing and connecting multiple representations for volume measurement is important

Task 3: Connecting the drawing vs. building task to standards

- Units: Unit cubes
- Task representation: 3D and 2D

Connections to CCSSM

Grade 5
Measure volumes by
- Viewing 3D shapes as decomposed into layers of arrays of cubes

Relate volume to multiplication and addition and solve real world and mathematical problems

Test ourselves: An opportunity to practice taking anecdotal notes

As you watch, record key aspects of what you see and hear on the anecdotal notes template for volume

Volume Quantity Recognizer (VQR) *Filling and Packing.* Recognizes capacity as an attribute. *Building.* Builds with blocks, associating more blocks with terms like "big" and fewer blocks with "small" *Comparing.* Describes with words such as big, small, and tiny. Compares "volume" using only one dimension. **Volume Filler (VF)** *Filling.* Fills a container using smaller container, counts the number needed to completely fill the larger container. *Packing.* Puts cubes into rectangular box to fill - may leave gaps; eventually, packs entire box leaving no gaps. *Building.* Recognizes and counts cubes (the child may be counting "blocks" or even "squares") on multiple faces. *Comparing.* Compares objects by physically or mentally aligning; refers to a least two dimensions of objects. 1	**Volume Quantifier (VQ)** *Filling.* Estimates number of scoops needed to fill. Attends to both portion filled and portion remaining unfilled. Recognizes when container is half full. *Packing.* Packs box neatly and completely with cubes; counts one cube at a time, while packing. *Building.* Builds without gaps. Counts on all faces of an object constructed of cubes, has a developing sense of the cube as a unit, begins to recognize that squares on adjacent faces of a rectangular prism sharing a side are faces of the same cube. *Comparing.* Compares the volume of objects by counting the number of cubes or by physically or mentally aligning and explicitly recognizing three dimensions. 2	**Volume Unit Relater and Repeater (VURR)** Relates size & number of units explicitly; understands that fewer larger than smaller units will be needed to fill/pack a given container. Can accurately convert units in 1:2 ratio. *Filling.* Uses simple units to fill containers with accurate counting, completely filling scoop each time. After one unit has been poured into the container can anticipate the volume of the container by iterating. *Packing.* Packs completely using discrete units with accurate counting. Iterates unit throughout volume, maintaining equal unit size & spacing. *Building.* Exhibits developing understanding of cubes as filling space. Counts cubes, not faces. *Comparing.* When comparing two 3-D objects, describes correctly the relative volumes of objects by reasoning about unit size. 3
Initial Composite 3-D Structurer (VICS) Understands cubes as filling a space. Explicitly relates size and number of units to volume. Uses additive reasoning. *Filling.* Relates number of cubes to cubic units as measured by capacity. *Packing.* Begins to visualize and operate on composite units (rows or columns). Iterates accounting for "internal/hidden" cubes. Decomposes space with accurate use of units/subunits. *Building.* Develops more accurate counting strategies. Counts systematically, accounting for internal/hidden cubes, and moves to operating on composites. *Comparing.* Connects volume as number and volume as space. Develops sense of conservation for the cases in which transformation is involved. 4	**3-D Row and Column Structurer (VRCS)** Able to flexibly coordinate *filling, packing, building* aspects of volume. Uses additive comparisons (e.g., "this one has 12 more") but may show some multiplicative comparisons (e.g., "this one is four times as big"). Initially counts or computes the *number of cubes in one layer* then uses addition or skip counting by layers to determine total volume; eventually moves to multiplication. Operates fluidly and flexibly on units (cubes), units of units (rows or columns), and units of units of units (layers). Composes and decomposes array ←→ layers ←→ rows/columns ←→ units. With perceptual support, can decompose 3-D arrays into other, complex 3-D arrays (not only layers, rows, or columns) and calculate the number of these smaller arrays in the larger array. 5	**3-D Array Structurer (3D AS)** Has an abstract understanding of the rectangular prism volume formula. Coordinates multiplicative and additive comparisons flexibly. With linear measures or other similar indications of the three dimensions, multiplicatively iterates cubes in a row, column, and/or layer to determine volume. In multiple contexts, can compute the volume of rectangular prisms from their dimensions. Visualizes and operates on both horizontal and vertical layers, even without perceptual support. Decomposes 3-D arrays into other complex 3-D arrays (not only layers, rows, or columns) and calculates the number of smaller arrays in the larger array (by using repeated addition, multiplication, or division). Coordinates the spatial and symbolic decompositions. 6

LT code	Evidence (Notes/Images)

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Test ourselves 1

Test ourselves 1

3-D Row and Column Structurer

Test ourselves 2
(1st grade)

Volume Quantifier

Test ourselves 3
(Same boy, 2^{nd} grade)

Initial Composite 3-D Structurer

Summary

In this session you:

- Analyzed examples of student engagement in measurement in terms of the learning trajectory for volume measurement
- Used learning trajectory levels to predict performance on example tasks