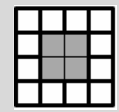
Math Notes: Pool Border Problem

Description of the task:

How many square tiles does it take to build a border around a square "pool"?

Find a way to know the number of tiles it will take, without having to count, for any size pool.[1]



When solving this problem, it is easy to make simple visual representations (using graph paper or tiles) such as the figure above, which shows a pool of side length 2. These representations make it possible to experiment with small pool sizes—and a variety of counting strategies in general—using various decompositions of the border. These in turn lead to a variety of algebraically equivalent formulas for the general solution. Since the number of border tiles equals its area, the problem also exhibits links between perimeter and area.

What approaches could be used when working on the problem?

There are numerous ways to generate a representation for finding the number of square tiles in the border of a square pool that has side length s. For each particular representation, an algebraic expression can be generated, all of which are equivalent to $4s + 4$. These different ways can be categorized into three general approaches to the problem: linear, tabular/experimental, and area. Each of these approaches is discussed below and example(s) are provided.

[1] It is tacitly to be understood that the side length of the pool is a whole multiple of the unit length.

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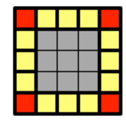
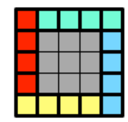
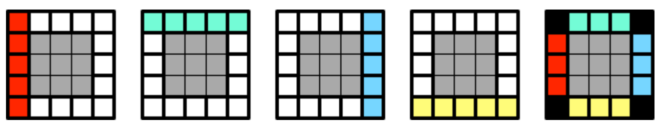
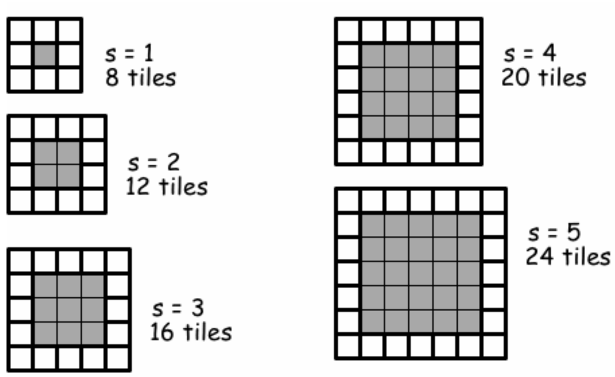
Approach #1: Linear

A linear approach entails decomposing the border of the pool into square tiles. These tiles can be arranged in a straight line and the length of the line can be determined. Because the side length of each square tile is 1 unit, the number of tiles in the border can be computed directly from the computed length (e.g., 16 square units requires 16 tiles). The border of the pool can be decomposed into linear segments in many different ways, all of which lead to an algebraic expression representing the number of tiles needed for the border equivalent to 4s + 4. The table below illustrates three such approaches. Each approach shown is based on a square pool with a side equal to 3 units (s = 3) and a border of 16 square tiles.

Approach	Description, algebraic expression, and picture
a	The number of tiles needed to cover the perimeter of each side of the pool, plus 4 tiles for the corners. <u>This case:</u> 4(3) + 4 <u>General:</u> 4s + 4 
b	For each side of the four sides of the border, the number of tiles needed to cover the perimeter of the pool plus one tile per side to cover one corner. <u>This case:</u> 4(3 + 1) <u>General:</u> 4(s + 1) = 4s + 4 
c	The number of tiles needed for the border of each side minus 4 tiles because each corner gets counted twice with this method.  <u>This case:</u> 4(3 + 2) – 4 <u>General:</u> 4(s + 2) – 4 = 4s + 4

Approach #2: Tabular/experimental

A tabular/experimental approach involves computing the borders for pools that are small in size, organizing such data in a table, and then looking for a numerical pattern. Typically, this approach starts with experimentation with pools of various sizes. The side length and the number of tiles for each such pool can be directly counted.



This approach can be extended by representing the data in a table and looking for a numerical pattern. One such table, summarizing the data from the experimentations above is illustrated below.

Side of pool	Number of tiles in the border	
1	8	4
2	12	4
3	16	4
4	20	4
5	24	

The table shows that the growth is constant (i.e., for each increase of 1 in the side length, the number of tiles in the border increases by a fixed amount). The number of tiles in the border can then be interpreted as a function of the side length of the pool:

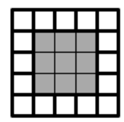
$4s + 4$ where s is the length of the side of the pool

Approach #3: Area of the border

A third approach entails determining the area of the border, which is the difference between the area of the two squares.

Area of the bigger square (pool + border) minus the area of the smaller square (the pool)

Because the border is composed of square tiles, each with an area of 1 square unit, the area directly shows the number of tiles needed (e.g., a border with an area of 16 square units requires exactly 16 square tiles). In the case of a square pool with a side equal to 3 units (s = 3) and a border of 16 square tiles:



$5^2 - 3^2$

More generally, the approach leads to the expression: $(s+2)^2 - s^2$ which is equivalent to $4(s + 1)$.

What are common ways that students approach this problem?

Variations of this task can be used throughout the elementary grades.

1. Kindergarten – Grade 2: When this task is used in the primary grades, children often engage in the experimental portion of the experimental/tabular approach. In other words, children are calculating the number of tiles needed for the border of various sized square pools by drawing the pools and counting the number of tiles in the border. One challenge is helping children understand the idea of a pool with a border. Graph paper can be used as a support for this work. Some primary grade students may find a short cut to counting the number of square tiles that parallel the algebraic expressions listed above.
2. Grades 3-5: In the upper elementary grades, students will often use experimental approaches as well as linear approaches. Experimental approaches often lead to linear approaches as students begin to notice ways to more quickly count the number of tiles in the border. When an experimental approach is used, children may need support in moving to a tabular representation, which supports the noticing of patterns. Across these different approaches, children often need support in moving from identification of tiles for a particular case to the identification of generalized expression that can be used to generate the number of tiles for a square pool of any size. A related challenge for upper elementary school students is seeing whether or not a method works for any sized square pool.

What mathematical practices are particularly relevant for working on this problem?

The Pool Border Problem provides opportunities to experience each of the eight mathematical practices. We focus here on the four practices that are most relevant to work on this problem.

MP1. Make sense of problems and persevere in solving them.

Successful engagement with the Pool Border Problem requires understanding what the problem is asking and looking for entry points into the solution (opportunities to make sense of problems). A common strategy, and one that is useful in persevering in solving the problem, involves examining analogous cases (the results of pools of various sizes) before making a generalization about how to find the number of tiles in the border of any square pool. It also involves explaining the correspondence between a representation of a particular pool and the rule/formula presented.

MP2. Reason abstractly and quantitatively.

A major component of this work is creating representations of the general problem and of passing from patterns in concrete calculations of small cases to general algebraic expressions. In developing generalized formulas for the border and showing their equivalency, the problem needs to be abstracted from the context of the problem. Interpreting the answer, though, requires a reconsideration of the context of the problem.

MP3. Construct viable arguments and critique the reasoning of others.

Successful engagement with this problem entails constructing arguments about how to find the number of tiles needed to build a border around a square pool of any given size. There is the opportunity to learn to analyze situations (pools of different sizes) by breaking them into cases. Conjectures can be made and represented. Building a logical progression of statements to explore the truth of their conjectures is entailed in this work. There are also opportunities to compare the effectiveness of two plausible arguments by listening to others' explanations of how they developed their formula/rule.

MP7. Look for and make use of structure.

Mathematical structure figures in each of the approaches to the problem discussed above. In the linear approach, the decomposition of the border as an assemblage of linear pieces is a geometric structure, on which the counting is based. In the tabular/experimental approach, the numerical pattern is a structure in the data. In the area approach, area is a geometric structure used in place of the combinatorial structure of tile counting. Finally, showing the equivalence of the various formulas produced makes use of the structure of algebraic expressions and their allowable transformations.