Math Notes: The Three-Coin Problem

Description of the task:

I have pennies, nickels, and dimes in my pocket.

If I pull out 3 coins, what amounts of money might I have?

This is a combinations problem that requires determining which amounts of money can be generated by selecting three coins from a collection of pennies, nickels, and dimes. In this problem, it is understood that there are many coins of each type in the pocket (at least three of each kind). A first stage of engagement with the problem is to produce one or more solutions (i.e., amounts of money that could be pulled out of the pocket). A second stage is to try to find all possible solutions. There are ten solutions in all (3¢, 7¢, 11¢, 12¢, 15¢, 16¢, 20¢, 21¢, 25¢, 30¢). A final stage is to then prove that all of the solutions have been found. In the course of this work, it is important to find a structured way to record and organize the set of solutions found; this is a complex representational task.

This problem can be rescaled in various ways to increase or decrease complexity, thus making it suitable for different grade levels. For example, one can vary the number of coins that are being pulled out of the pocket. Also, one can vary the number of types of coins and their denominations (e.g., by including quarters or by using pennies, dimes, and dollars in order to make connections to place value). By rewording the problem to ask for the different combinations of three coins that could be made (instead of asking for amounts of money), the problem can be rescaled to be a "pure" combinations problem. Asking for the amounts of money adds an arithmetic component to the problem.

What approaches could be used when working on the problem?

There are multiple approaches that can be used to work on this problem. Three of these approaches are: (1) engaging in random repeated trials; (2) counting ordered lists of three coins; and (3) finding three of a kind, two of a kind, and one of kind.

Approach #1: Random Repeated Trials

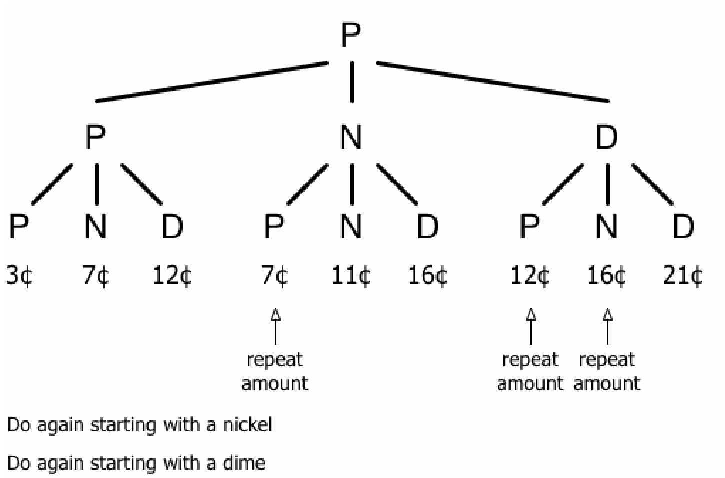
One approach, illustrated below, is to consider the different combinations that meet the conditions of the problem. Collections of coins, or permutations, are generated at random (e.g., reaching into a bag and pulling out three coins). These permutations are then recorded and their monetary values are determined. Generally, this approach continues until one has a sense that all of the solutions have been found (e.g., when further efforts continue to yield solutions that have already been found). The repeated trials approach often results in duplicate combinations. This approach, without further structure in the method of recording solutions, does not result in a proof that all of the solutions have been found.

PPP = 3¢	DPD = 21¢	<u>Amounts</u>
PND = 16¢	NND = 20¢	3¢
DPN = 16¢	PPP = 3¢	7¢
DNN = 20¢	PPN = 7¢	11¢
NPP = 7¢	NNP = 11¢	16¢
PNN = 11¢	NDP = 16¢	20¢
NPP = 7¢	NNP = 11¢	21¢
PPN = 7¢	NPP = 7¢	
DPN = 16¢		
DDP = 21¢		
PDN = 16¢		

School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • mtlt@umich.edu

Approach #2: Counting ordered lists of three coins

In this approach, one considers the coins to be pulled in a particular order: first coin pulled, second coin pulled, third coin pulled. Since there are three choices (penny, nickel, dime) for each of the coin pulls, there will be $3^3 = 27$ possibilities altogether. These can be represented in a tree diagram, illustrated below for the cases when the first coin pulled is a penny. Since the monetary value of the three coins does not depend on the order in which they are pulled, this method will produce many duplicate values. This approach is time-consuming and results in many duplicates, but it does support a proof that all possible amounts are included (many more than once).



Approach #3: Finding three of a kind, two of a kind, and one of a kind

This approach focuses on identifying the three different cases for the combinations of coins that can be drawn from the pocket—three of a particular coin (e.g., 3 pennies), exactly two of a particular coin and one of a different coin (e.g., 2 pennies and 1 dime), and three different coins (i.e., a penny, a nickel, and a dime). In effect, the problem is broken into three smaller problems—the three different cases of coin combinations—and the amounts that can be generated through each of these cases are considered. This approach is illustrated below. For each case, all of the combinations that are possible for the case are identified, and the resulting amounts are determined. When using this approach, a table may be used to organize the findings. Using this approach is an efficient way to determine all of the possible cases and the amounts possible. No duplicate solutions are generated, and it is possible to justify that no additional amounts are possible, given the conditions of the problem.

(1) Three of a kind		
	PPP	3¢
	NNN	15¢
	DDD	30¢
(2) Two of a kind		
	PPN	7¢
	PPD	12¢
	NNP	11¢
	NND	20¢
	DDP	21¢
	DDN	11¢
(3) One of a kind		
	PND	16¢

What are common ways that students approach this problem?

Throughout the elementary grades, students often begin solving this problem (as well as versions that are scaled to work in the lower elementary grades) by using approach #1, repeated trials. Some students may be convinced using this approach that they have found all of the possible amounts; however, such an approach does not lead to a proof. In initial work on the problem, it can be quite powerful for students to discuss possible amounts and why they are possible amounts by using the conditions of the problem[1], without considering the proof for justifying that all of the amounts have been found. This is because children will often generate solutions that are not solutions to the problem (e.g., a solution of 27¢ by considering one quarter and two pennies).

Throughout the elementary grades, children can be supported in generating representations such as the ones shown in approaches #2 and #3 to consider possible amounts. Tree diagrams are frequently familiar to children as they are often used in finding the prime factorization of whole numbers and in instruction on probability to determine all of the possible outcomes.

What mathematical practices are particularly relevant for working on this problem?

Work on the Three-Coin Problem can provide opportunities to engage in multiple mathematical practices. Three focal practices are identified below, and ways in which the engagement is supported by the problem are described.

MP1. Make sense of problems and persevere in solving them.

> Understanding the conditions and constraints of the problem, which coins and how many of them can be used, is foundational to the task. It is particularly important to be able to identify how one knows that a particular solution is a solution to the problem and eventually to find all possible amounts and to prove that all amounts have been found.

MP3. Construct viable arguments and critique the reasoning of others.

> Viable arguments need to be constructed for both individual cases (e.g., why 21¢ is a solution) and in justifying the conclusion that all possible solutions have been found. As there is more than one way to show that all solutions have been found, listening to and critiquing others' reasoning is important. In addition, students need to know that stating that no one can come up with another amount is not a sufficient justification that all solutions have been found.

MP7. Find and use mathematical structure.

> In order to make a convincing argument that all possible amounts of money have been found, the solutions to the problem must be organized in a structured way.

[1] The conditions of this problem include: 1) three coins must be pulled; 2) the coins must be pennies, nickels, and/or dimes; 3) the monetary value of the coins must be determined.