Math Notes: The Sum of Consecutive Odd Numbers Problem

Description of the task:

> What is the sum of the first 10 consecutive odd numbers greater than zero?
>
> Find a way to know the sum of any set of consecutive odd numbers (that start with 1) without adding every number in the set. Why are you convinced that this way will always work?

The first part of this task requires identifying the first 10 consecutive odd numbers greater than 0 and then finding the sum of the set of numbers:

The first 10 consecutive odd numbers are: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

The second part of the task involves making a generalization about how to find the sum of any set of consecutive odd numbers that start with 1. This task provides an opportunity to use inductive reasoning, which is the process of making a generalization from one or more specific cases. Using inductive reasoning entails noticing patterns in specific cases (e.g., noticing patterns when finding the sum of the first ten consecutive odd numbers starting with 1) and then using those patterns to develop ideas that lead to a general method (e.g., a method for finding the sum of any set of consecutive odd numbers starting with 1).

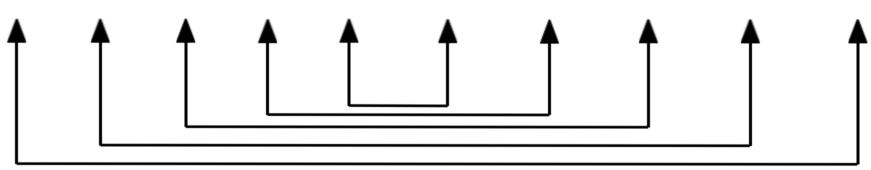
What approaches could be used when working on the problem?

1. One way to find the sum of 10 consecutive odd numbers beginning with 1 is to add these numbers together. This approach may or may not produce ideas that lead to finding the general case.

   Adding the numbers in consecutive order: If the sum is found by adding in order, then there is no path to the general case.

   Adding after reducing the number of addends by pairing numbers: If addition is seen as reducing the number of addends by pairing numbers, then the general case can evolve. When pairing addends, the question to be posed is whether there is a way to pair the numbers such that each pair results in the same sum. If this is possible, then multiplication can be used to find the sum of all the numbers. Such a pairing is:

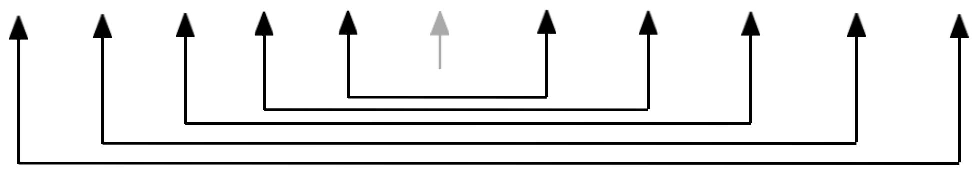
$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$$



   In this pairing, each pair of two numbers has a sum of 20. There are five pairs of 20, or a sum of 100, and the number of pairs is half of the amount of consecutive numbers listed. These insights will be useful for generalizing about sums of consecutive odd numbers beginning with 1.

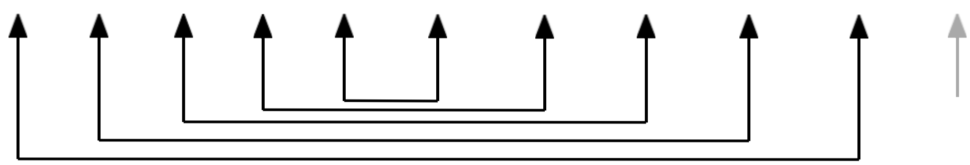
To generalize this method for finding the sum of a set of consecutive odd numbers for any set of consecutive odd numbers beginning with 1, it is important to see if it works for sets that contain an odd number of numbers. For example, when summing a set of 11 consecutive odd numbers beginning with one, pairing the numbers starting with the outside numbers and working inward yields five pairs of numbers with a sum 22 plus one number (11) left over.

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$$



Similarly, pairing from the middle yields five pairs of 20 with one number (21) left over.

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$$



The sum in both approaches is the same $5 * 22 + 11 = 5 * 20 + 21 = 121$.

To get to a general approach, it is helpful to look at shorter sequences to see which, if any, patterns exist. When finding the sum of the first four consecutive numbers, 1 + 3 + 5 + 7, there are two pairs with sums of 8 and adding these eights yields a sum of 16. When finding the sum of the first five numbers, 1 + 3 + 5 + 7 + 9, there are two pairs with sums of 10 plus a 5 that remains after the process of pairing is complete. Adding 10 + 10 + 5 produces a sum of 25.

A couple of different generalizations can be made based on these patterns. One generalization relates to the process of finding sums by making pairs. We see that the number of pairs equals half the number of numbers. For example, with 11 consecutive odd numbers, that is (11 ÷ 2 or 5.5).

Multiplying that number of pairs by the sum of the first and last number will always yield the sum of the set of consecutive odd numbers.

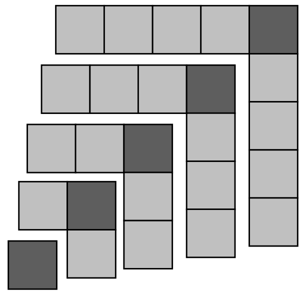
Some may notice that the sum is a square number. When we sum the first 10 numbers, the sum was 100; of the first 11 numbers, 121; of the first four numbers, 16; the first five numbers, 25. A possible generalization would be that the sum is the square of the number of numbers. A geometric approach that illustrates this insight is described below in approach 3.

2. Another approach to this problem could come from recognizing that the sequence of odd numbers beginning with 1 is an "arithmetic sequence". For the specific sequence, there are 10 numbers, the first is 1 and the last is 19 and the difference between the numbers is two. The formula for the sum of the arithmetic sequence or progression is $(\frac{n}{2})(a + l)$, where n is the number of numbers, a is the first term and l is the last term.

$$\frac{n}{2}(a + l) = \frac{10}{2}(1 + 19) = 5 * 20 = 100$$

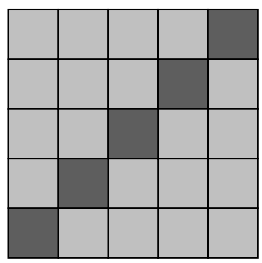
One way of reasoning about why this approach works is described above in approach 1.

3. In the first discussion of finding the sum of the sequence of odd numbers beginning with 1 by adding we noted that the sum is n^2, where n is the number of numbers. We can use a geometric representation of odd numbers to provide a convincing argument that n^2 should be the sum. One representation of the odd number seven is



1 → 3 5 7 9

where the two groups of the three lighter gray squares are separated by one darker gray square. We can use this representation to build an actual square, thus illustrating that the sum is a square number with a clear relationship to the given number of consecutive odd numbers.

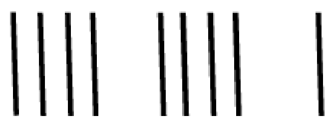


$1 + 3 + 5 + 7 + 9 = 5^2 = 25$

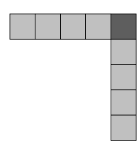
What are common ways that students approach this problem?

To solve this task, students need to know what odd numbers are. Students may have different ways of thinking about this. One way of thinking about an odd number is as a number that can be made with groups of two leaving one that is not in a group of two. This could be represented symbolically as $n * 2 + 1$. For example, the odd number 9 could be seen as having 4 groups of 2 + 1. A representation of this interpretation could be:

Another way of thinking about an odd number is as 2 groups of a given number of objects plus 1, then the symbolic representation is $2n + 1$. A representation of that idea could be:



Another representation of an odd as two groups of the same number plus one is:



In this picture, the two equal groups (e.g., two groups of 4) are represented by the horizontal row and vertical column of light gray squares, and the one left over is represented by the darker square connecting the row and the column. As illustrated above in approach 3, this can be a helpful representation to use when solving the Sum of Consecutive Odd Numbers Problem.

It is mostly likely that students will first approach this problem by adding the numbers in the sequence that numbers are written. While it is possible to determine the sums of particular sets of consecutive numbers in this way, it is unlikely to lead to a general solution. With encouragement, students can be prompted to look for short cuts in the addition, for patterns in the sums they find, and/or for relationships between particular sums and the number of consecutive numbers that were added. These explorations can lead to versions of approach 1 as explained above. While much less common, it is possible that supplying students with grid paper or with square tiles could support them in building odd numbers and perhaps in constructing the geometric representations of the sums of particular sets of consecutive odd numbers. This would position them for visualizing approach 3 as explained above.

What mathematical practices are particularly relevant for working on this problem?

As students work on generating sums of consecutive odd numbers that start with 1, they have opportunities learn to engage in several of the mathematical practices including:

MP 2. Reason abstractly and quantitatively.

Discussions of making generalizations for the sum of consecutive odd numbers starting with 1 provide opportunities to talk about Mathematical Practice 2: "Reason abstractly and quantitatively." As students find the sums of a sequence of odd numbers starting with 1, they create representations that are used in their work. Both the representation of the pairing of the numbers in the sequence and the geometric representation of odd numbers as "L-shaped" objects that nest to make a square could be important parts of the solution to the task.

MP 7. Look for and make use of structure.

Discussions of odd numbers and the finding of the sums of such sequences provide opportunities to talk about Mathematical Practice 7: "Look for and make use of structure." As students create representations of odd numbers or the pairing of odd numbers in a sequence, they use the structure inherent in odd numbers, the two groups of equal size and an additional 1, or the structure of a sequence of odd numbers that enables them to pair numbers that equal the same sum. These structures are crucial to developing methods for finding the sum of any sequence of odd numbers starting with 1.

MP 8. Look for and express regularity in reasoning.

Discussions of finding of the sums of sequences of odd numbers starting with 1 provide opportunities to talk about Mathematical Practice 8: "Look for and express regularity in reasoning." When students pair the numbers in a sequence of odd numbers starting with 1, they repeat the calculation of a sum that is used in developing a formula for finding the sum of such sequences.