Math Notes: Odd + Odd = Even

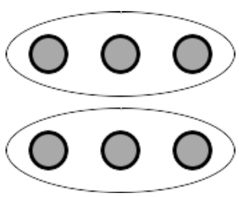
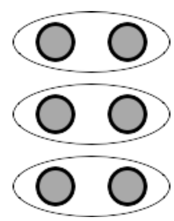
Description of the task:

Prove that the sum of any two odd numbers is an even number.

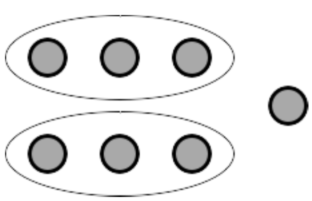
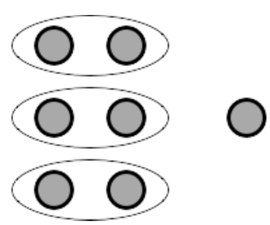
This task requires the use of definitions for odd and/or even numbers. The process of proving or demonstrating is likely to involve the development and use of representations of odd whole numbers.

Mathematical Background:

Even numbers are numbers that can be written as two times a whole number. Symbolically, an even number is a number that can be written as 2n, where n is a whole number. The expression 2n can be interpreted in two ways. It can be interpreted as 2 groups with n objects in each group. Through the use of the commutative property, it also can be interpreted as n groups of two objects each. In the figure below, these interpretations are represented using 6 as an example.

Even numbers interpretation #1: Two groups of n objects	**Even numbers interpretation #2: n groups of two objects**
6 can be interpreted as 2 groups with 3 objects in each group 	6 can be interpreted as 3 group with 2 objects in each group 

Odd numbers are numbers that can be written as two times a whole number plus 1. Symbolically, an odd number is a number that can be written as 2n + 1. As with the definition of even numbers, the expression 2n + 1 can be interpreted in different ways. In the figure below, these interpretations are represented using 7 as an example.

Odd numbers interpretation #1: Two groups of n objects plus 1	**Odd numbers interpretation #2: n groups of two objects plus 1**
7 can be interpreted as 2 groups with 3 objects in each group plus 1 more 	7 can be interpreted as 3 group with 2 objects in each group plus 1 more 

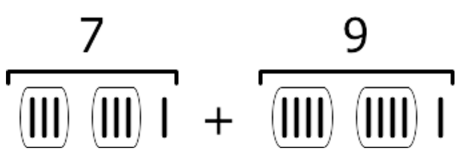
To prove that the sum of two odd numbers is always an even number, one needs to build from these understandings of odd and even numbers which allow one to show that the conjecture is true for specific cases in order to generate a way to prove that the statement is generalizable.

What approaches could be used when working on the problem?

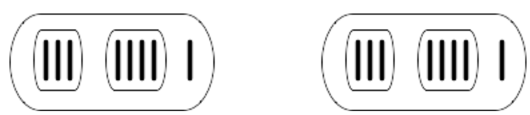
There are multiple approaches that can be used to work on this problem. Three of these approaches are: (1) a geometric approach using the first interpretation of an odd number; (2) a geometric approach using the second interpretation of an odd number; and (3) an algebraic approach using either interpretation.

Approach #1: Geometric approach using the interpretation of odd numbers as two groups with n objects in each group plus 1.

If odd numbers are represented as two groups with n objects in each group plus one additional object, then the sum of two odd numbers can be represented geometrically by drawing any two odd numbers with that configuration. We will use the example of 7 + 9 to illustrate this approach. Below, the number 7 is represented by two groups of 3 plus an additional 1 and the number 9 is represented by two groups of 3 plus an additional 1.



Combining one of the equal sized groups from 7, one of the equal sized groups from 9, and a left over 1 forms a new group of size 8. The objects left over after that process can also be made into a group of size 8.

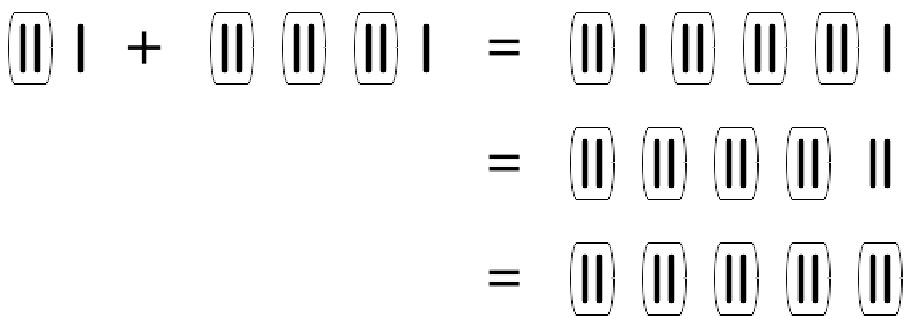


The two equal-sized groups of 8 demonstrate that this number, 16, is an even number.

Using representations to prove that the sum of any two odd numbers is an even number is challenging because representations could be viewed as illustrating specific cases instead of the general case. To prove that this is generalizable, it is necessary to articulate that when making other numbers one would add or subtract the same number of objects to/from each of the two equal-sized groups, the structure of the number would remain the same – the groups remain equal-sized – and there is one additional object. Hence, adding two odd numbers allows the extra 1 in each number to be paired, illustrating that the sum is even.

Approach #2: Geometric approach using the interpretation of odd numbers as n groups of two objects in plus 1.

If odd numbers are represented by groups of 2 objects plus one additional object, then the sum of two odd numbers can be represented geometrically by drawing two odd numbers with that configuration. We will use the example of 3 + 7 to illustrate this approach. Below, the number 3 is represented as one group of 2 objects plus 1 additional object and the number 7 is represented as three groups of 2 objects plus 1 additional object. Adding the two odd numbers allows the extra object (the 1 leftover after making groups of size two) in each number to be paired, illustrating that the sum is even.



In order to demonstrate that this is generalizable it is necessary to articulate that adding more groups of two or subtracting some groups of two does not change the structure of the number. In other words, for any specific case of odd number there is a given number of groups of two with one extra object.

Approach #3: Algebraic approach using the interpretation of odd numbers as n groups of two objects with one object left over.

An algebraic approach can be done using either of the interpretations of odd numbers described in the mathematical background section. Here, the definition of an odd number as n groups with two objects in each group plus 1 used. Any two odd numbers can be written as follows:

Odd number #1: $2a + 1$, where a is a whole number

Odd number #2: $2b + 1$, where b is a whole number

The sum of these odd numbers is:

$$\begin{aligned}(2a + 1) + (2b + 1) &= 2a + 1 + 2b + 1\\ &= 2a + 2b + 1 + 1\\ &= 2(a + b) + 2\end{aligned}$$

The sum of $2(a + b) + 2$ can be interpreted as $(a + b)$ groups containing two objects with 2 objects leftover which make one additional group. Algebraically, when adding two odd numbers, we sum the number of groups of size 2 from each of the odd numbers and the "one left over" in each of the odd numbers comes together to form an additional pair.

What are common ways that students approach this problem?

Students tend to use the geometric strategy shown, but they often differ with respect to the interpretation of odd number (two groups + 1 vs. groups of two + 1) that they use. Often multiple examples are shared that may be accompanied by representations showing the addends and the sum. Recall that what is key to accepting such demonstrations as "proofs" that work no matter what odd whole numbers are chosen is the articulation of how the structure of the representations of the two numbers will remain the same as other examples using the same interpretation.

It may be difficult for students in the primary grades to articulate the generalization even if they understand the structure of odd numbers. Prior to working on the given conjecture, students may benefit from experience with concrete materials representing odd and even numbers. One approach involves using a typical egg carton (six by two) to show even and odd numbers. An egg carton can be cut into different lengths of two by whatever number to represent even numbers. A single egg carton piece can be used to show what an odd number is – an even number plus 1 no matter what the even number is.

Add two odd numbers means that you have two single egg compartments that can be paired to make a set of two, making the number even.

Alternatively, one-inch square graph paper, cut into two rows by whatever number pieces you want for the length also can be used. Attaching a single square to any even number (two rows of squares) can be done to make an odd number. Again, the matching up of two odd numbers shows that the lengths of the rows of squares does not matter, there is always one more pair that can be made when the two single squares are matched.

What mathematical practices are particularly relevant for working on this problem?

As students justify the conclusion that the sum of two odd numbers is always an even number, they have the opportunity to engage in several mathematical practices including:

MP 1. Make sense of problems and persevere in solving them.

> Students often analyze the given information and connect the problem with definitions of what it means for a number to be odd or even. They persevere through multiple experiments to see if the statement is actually true and then with examples, representations, and ways of thinking in order to arrive at a generalization that could hold for all cases.

MP 2. Reason abstractly and quantitatively.

> Students often create representations of odd and even numbers to facilitate their thinking and to support later demonstrations to classmates. Students reach beyond particular numerical values and ponder how ideas in particular cases illustrate what would happen within an infinite set of whole numbers.

MP 3. Construct viable arguments and critique the reasoning of others.

> Students often construct viable arguments using definitions of odd and even numbers, justify their conclusions while communicating their arguments to others, and reason inductively about examples. Often, they will need to consider the merits and reasonableness of interpretations of odd and even numbers other than the numbers that they use in the cases that they explore.

MP 7. Look for and make use of structure.

> Students can use structure to develop arguments that are more than just a collection of examples that support the conclusion that the sum of two odd numbers is an even number. Odd whole numbers build on the structure of even whole numbers in their definition or representation. The inherent difference between even and odd numbers is the presence of an additional object or the number 1 in odd numbers beyond the groups of two objects or the equal-sized groups of even numbers.