Math Notes: Justifying Geometry Statements

Being able to justify whether a mathematical statement is true or false is an important part of the work of mathematics that students can engage in throughout the grade levels and across many mathematical topic areas. Justifying statements in mathematics may require:

- Using definitions, postulates, and theorems
- Using logic statements
- Identifying "hidden" or assumed quantifiers

Although students at different grade levels express their justifications differently, at all grade levels, students can learn to justify statements in ways that are appropriate and convincing for the mathematical communities in their classrooms. As students create justifications for mathematical statements, they have opportunities to learn to engage in several mathematical practices.

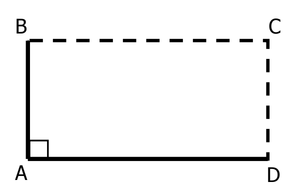
Using definitions, postulates, and theorems

To determine whether a mathematics statement is true or false, one needs to start with what is known about the object(s) under discussion. Three types of ideas form the basis of what a mathematical community knows about an object. They are:

- definitions (explain the meaning of a term that is agreed upon in a given community);
- postulates (ideas that the community assumes to be true without proving the ideas);
- and theorems (ideas that have been proven in the community).

When justifying mathematical statements, it is important to start by stating what the community knows about the object under discussion. What is known may change from classroom to classroom and from grade level to grade level, depending on what particular classroom communities have previously discussed and worked on. As students progress through the grades, definitions become more general and postulates may become theorems. To determine whether a statement is true or false, one must build a chain of reasoning to justify the conclusions that are being made. This chain of reasoning starts with the definition of the object and states how the definition relates to the statement under discussion. Then, relevant postulates and theorems about that object are applied, which lead to the conclusions that are being made.

Consider the statement, "Any parallelogram with at least one right angle is a rectangle." The illustration below shows an example that corresponds with the statement.



To prove whether the statement is true or false, the following process could be used:

Step 1: Stating the definitions of the mathematical objects (i.e., parallelogram and rectangle) included in the statement:

> **Parallelogram**: A 4-sided polygon whose opposite sides are parallel. The opposites sides of a parallelogram are the same length. Adjacent angles in a parallelogram have a sum of 180 degrees.

Rectangle: A parallelogram with four right angles.

Step 2: Using a chain of reasoning to build a justification for the statement that a parallelogram with at least one right angle has to have four right angles making it a rectangle.

1) We are told in the initial statement that the parallelogram has one right angle. Angle A is that right angle (and has a measure of 90 degrees as a result).
2) Angles A and B are adjacent angles along side AB. Based on the given definition of a parallelogram, Angle B must be a right angle because adjacent angles in a parallelogram have a sum of 180 degrees.
3) Similarly because Angles A and D are adjacent angles along side AD, Angle D is also a right angle.
4) Therefore, the remaining angle, Angle C must also have a measure of 90 degrees because both Angles B and D are adjacent with it.
5) **Conclusion:** Given a parallelogram with one right angle, the other three angles must also be right angles. This makes the parallelogram a rectangle because a rectangle, by definition, is a parallelogram with four right angles. Therefore, the given statement is true.

By following this type of process and ensuring that each step in the chain of reasoning follows from the previous statement, and employs information that is available to members of a particular mathematical community, valid justifications for statements can be constructed.

Interpreting mathematical statements

Mathematical statements can take many different forms, including "if-then" statements. Because statements can be written in many other forms, the implications of the statements are not always be clear. Sometimes, rewording statements as "if-then" statements can help make the underlying logic of the statement clearer. This can strengthen justifications of mathematical arguments. For example, simple declarative statements can be written as "if-then" statements to surface important relationships that might otherwise be overlooked.

For every declarative statement, there are three accompanying statements, the converse, the inverse, and the contrapositive. The table below includes the four types of related logic statements written both in simple declarative form and "if-then" form. The table is followed by an elaboration of the logic of each kind of statement.

Types of statements	Example statements	Examples of conditional "if then" form
Initial statement	A square parallelogram.	If a shape is a square, then it is a parallelogram. (If p, then q.)
Converse	A parallelogram is a square.	If a shape is a parallelogram, then it is a square. (If q, then p.)
Inverse	A shape that is not a square is not a parallelogram.	If a shape is not a square, then it is not a parallelogram. (If not p, then not q.)
Contrapositive	A shape that is not a parallelogram is not a square.	If a shape is not a parallelogram, it is not a square. (If not q, then not p.)

Converse. Converses are created by switching the hypothesis and the conclusion of a declarative statement. If a statement is true, the converse of the statement is not necessarily true. For example, although the statement "If something is a cat, then it is an animal" is true, the converse of the statement "If something is an animal, then it is a cat" is not necessarily true. In the table above, the converse of the statement "A square is a parallelogram" is "A parallelogram is a square", which is not true in all cases. A common error in reasoning is to assume that if a statement is true, then the converse of the statement must also be true.

Contrapositive. Contrapositives are created by switching the hypothesis and the conclusion of a statement and then negating both. For example, the statement "If something is a cat, then it is an animal" as a contrapositive would be written as "If something is not an animal, then it is not a cat." If a statement is true, then the contrapositive of the statement is also true.

Inverse. Inverses are created by negating both the hypothesis and the conclusion of a statement. For example, the inverse of the statement "If something is an animal, then it is a dog" is "If something is not an animal, then it is not a dog". If a statement is false, the inverse of the statement is not necessarily false. In the example, the given statement is false, but the inverse happens to be true. If the converse of a statement is true, then its inverse is also true. For example, the declarative statement "If something is an animal, then it is a dog" is not true. However, both the statement's converse ("If something is a dog, then it is an animal") and the statement's inverse ("If something is not an animal, then it is not a dog") are true. Interestingly, the inverse is the contrapositive of the converse of a declarative statement.

Rewording statements in the "if-then" form and then considering the relationships between statements (e.g., initial statement, converse, inverse, contrapositive) can help clarify whether a mathematical statement is true or provide direction for how one might go about determining the truth of the statement.

Identifying "hidden" or assumed quantifiers

Another concern in mathematical reasoning is recognizing the scope of cases under consideration in a particular statement. Mathematical statements often use (or imply) quantifiers to signal whether they are referring to particular cases, some cases, all cases, or no cases. There are many different ways to express these ideas. The table below shows examples of statements that use different types of quantifiers and also indicates different ways of expressing each quantifier.

Type of quantifier	Mathematical statement using the quantifier	Different ways to express the quantifier
All cases	The sum of the interior angles of all triangles is 180 degrees.	Every Whenever Always
Particular cases	The interior angles of an equilateral triangle each have a measure of 60 degrees	At least one is There exists a For some, you can find a
Some cases	Some triangles are isosceles.	At least one is There exists a For some, you can find a
No cases	No triangle has four sides.	No There does not exist For none Never

Attempting to prove the statement "All polygons with four connected sides are quadrilaterals" often prompts discussions of quantifiers. What does four mean? Does it mean at least four, exactly four, or no more than four? Is a pentagon a quadrilateral because it has five straight connected sides, meaning it has at least four straight connected sides? There also could be a discussion of what connected to means. Does it mean that the four sides need to be connected only to each other or can there be an additional side that completes the connection? Does connected imply crossing at the end points or could they cross anywhere?

These questions do not have universal answers. Instead, each community that is discussing this statement must clarify what they mean by the particular words in the statement. Coming to a consensus about the meaning of terms and "hidden" quantifiers will ultimately determine whether the statement under consideration is true or false.

What mathematical practices are particularly relevant when justifying mathematical statements?

As students practice creating justifications for mathematical statements, they have opportunities to learn to engage in several of the mathematical practices including, but not limited to:

MP 1. Make sense of problems and persevere in solving them.

As students determine whether a particular statement is true or false, they must first make sense of the statement and then work to build a convincing mathematical justification for that statement.

MP 3. Construct viable arguments and critique the reasoning of others.

When students build a justification for a statement in the context of a classroom community, they must ensure that their justification seems viable to others in their community. Likewise, as students listen to others in the classroom community present justifications, they have opportunities to critique these justifications and to judge whether or not they are convincing.

MP 6. Attend to precision.

Students may raise questions about implied quantifiers or they may interpret the meaning of quantifiers quite differently from their classmates making it difficult to work collectively on mathematics. Discussions provide opportunities to determine the nature of the precision of language required by the community. Agreements can be made about how the community chooses to use numbers in statements (e.g., four means exactly four). A community might also agree that statements must be modified so that they explicitly include quantifiers before a conclusion can be proven true or false. It is key that subsequent work reflect the precision required by community agreements.