Math Notes: Pascal's Triangle

Description of the task:

Pascal's Triangle is an array of numbers that includes many patterns. The Pascal's Triangle Problem provides an opportunity to identify patterns and to explain how particular patterns are produced.

Pascal's Triangle Problem

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

What patterns do you see in the triangle?

What "rules" do these patterns follow?

If these patterns continue, what numbers would be in the next row?

How do you know?

Pascal's Triangle is named after Blaise Pascal, who discussed the array of numbers in a book that was published in 1654. Historical records show that the Chia Hsien, a Chinese Mathematician, and Omar Khayyam, a Persian mathematician, both published treatises based on this triangular array in the 11^{th} century. One of the first uses of Pascal's Triangle (in the 11^{th} century) was in determining the coefficients of a binomial expansion. If the expression (x + y) is raised to any power, its coefficients are the numbers in that row of the triangle[1]. There are many fascinating patterns within Pascal's Triangle and many interesting applications of those patterns. Pascal investigated patterns within the triangle and applied his insights when problem solving, especially in the area of probability. While Pascal's Triangle is an object that is used for work in algebra and probability in later grades, many of the patterns in the triangle are "discoverable" and accessible to students in the elementary and middle school grades.

What patterns could be noticed when working on this problem?

Many patterns exist in Pascal's Triangle, most of which happen in the rows or the diagonals in the triangle. In this document, rows will be counted starting with Row 0 (see below).

Row 0 ► 1
Row 1 ► 1 1
Row 2 ► 1 2 1
Row 3 ► 1 3 3 1
Row 4 ► 1 4 6 4 1
Row 5 ► 1 5 10 10 5 1

Rows of Pascal's Triangle

The diagonals will be counted starting with Diagonal 1 (see below).

Diagonal 1 ◢ ◣ Diagonal 1
Diagonal 2 ◢ 1 ◣ Diagonal 2
Diagonal 3 ◢ 1 1 ◣ Diagonal 3
Diagonal 4 ◢ 1 2 1 ◣ Diagonal 4
Diagonal 5 ◢ 1 3 3 1 ◣ Diagonal 5
Diagonal 6 ◢ 1 4 6 4 1 ◣ Diagonal 6
1 5 10 10 5 1

Diagonals of Pascal's Triangle

Note that, because Pascal's Triangle is symmetric (see Pattern 5 below), the diagonals that start with a number on the left-hand side of the triangle and go toward the right-hand side are exactly the same as the diagonals that start with a number on the right-hand side of the triangle and go toward the left-hand side.

Patterns in the triangle include, but are not limited to:

1. Every row starts and ends with a 1.

1
1 **1**
1 2 **1**
1 3 3 **1**
1 4 6 4 **1**
1 5 10 10 5 **1**

[1] In the rest of the document, the word triangle is used to refer the triangular shaped array of numbers in Pascal's Triangle. It is notable that the object under consideration is not a triangle in a conventional geometric sense.

2. Each row has one more number than the row above. For example, row 3 has 4 numbers. The next row, row 4, has 5 numbers.

1

1 1

1 2 1

Row 3 ► **1 3 3 1**

Row 4 ► **1 4 6 4 1**

1 5 10 10 5 1

3. Each number in the triangle is the sum of the two numbers above it.

   1 + 2 = 3

   6 + 4 = 10

1

1 1

1 2 1

1 **3** 3 1

1 4 **6 4** 1

1 5 10 **10** 5 1

4. The sum of every row is a power of 2.

   The sum of row 0 is $1 = 2^0$.

   The sum of row 1 is $1 + 1 = 2 = 2^1$.

   The sum of row 5 is $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$.

1 = 1

1 + **1** = 2

1 + **2** + **1** = 4

1 + **3** + **3** + **1** = 8

1 + **4** + **6** + **4** + **1** = 16

1 + **5** + **10** + **10** + **5** + **1** = 32

5. The array is symmetric. One can envision folding it vertically with a line passing through the middle of each row of numbers, resulting with numbers on one side of line that mirror the numbers on other side.

1
1 | 1
1 2 1
1 3 | 3 1
1 4 6 4 1
1 5 10 | 10 5 1

6. Each diagonal follows a different rule (recall that each diagonal appears twice, either starting from the left or the right side of the triangle). Here are some:

   Diagonal 1 contains all ones.

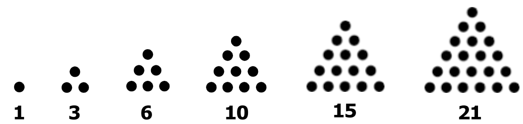
   Diagonal 2 contains the counting numbers (1, 2, 3, 4, 5).

1
1 1
1 **2** 1
1 3 **3** 1
1 4 6 **4** 1
1 5 10 10 **5** 1

   Diagonal 3 contains the triangular numbers.

1
1 1
1 2 1
1 **3** 3 1
1 4 **6** 4 1
1 5 10 **10** 5 1

   A triangular number counts the number of objects, such as dots, that can form an equilateral triangle. The first six triangular numbers are 1, 3, 6, 10, 15, and 21.



The differences between consecutive pairs of numbers in diagonal 3 increase by 1.

3 – 1 = **2**

6 – 3 = **3**

10 – 6 = **4**

1
1 1
1 2 1
1 **3** 3 1
1 4 **6** 4 1
1 5 10 **10** 5 1

The sum of two consecutive numbers in diagonal 3 is a square number. The sum produced is the square of the number in diagonal 2 that is in the same row as the lesser number in the pair of numbers that is being added in diagonal 3 For example, the 4 in the second diagonal is adjacent to 6 and 10 in diagonal 3. 4^2 is equal to the sum of these two numbers.

$4^2 = 6 + 10 = 16$

1
1 1
1 2 1
1 3 3 1
1 4 **6** **4** 1
1 5 10 **10** 5 1

7. In the odd rows, the middle numbers are repeated.

1
1 **1**
1 2 1
1 **3** **3** 1
1 4 6 4 1
1 5 **10** **10** 5 1

8. If the first non-1 in a row is a prime number, every non- 1 in the row is divisible by that prime. In Row 7, for example, each non-1 number is divisible by 7.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

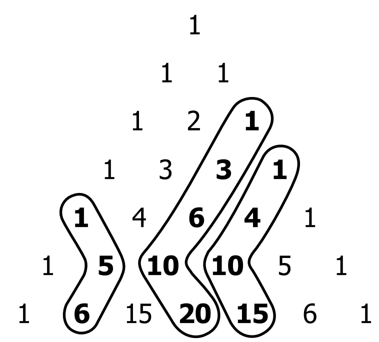
1 **7** **21** **35** **35** **21** **7** 1

9. For a diagonal of any length starting with 1, the sum of its numbers equals the number below the last number of the diagonal that is not on the diagonal. This is sometimes called the "hockey stick" or "sock" pattern.

   1 + 5 = 6

   1 + 3 + 6 + 10 = 20

   1 + 4 + 10 = 15



What are common ways that students approach this problem?

Students often identify "linear" patterns in the triangle: these patterns include patterns in the diagonals and patterns in the horizontal rows. Students will recognize diagonals that follow the same pattern such as noticing that the outermost diagonals that run from the top to the left and from the top to the right are the same.

Although students often notice the outermost diagonal of 1, 1, 1, … they may not believe that this representation is a pattern because the terms do not vary. Students will notice the counting numbers in the second diagonal and will describe the pattern in terms of counting. It is harder for students to recognize that diagonals beyond the second diagonal have patterns, let alone articulate the rules that generate those patterns. Explicit work to help students quantify the amount that each term changes from the previous term and then to notice patterns in the amount of change across multiple terms in a diagonal can provide a tool for noticing and explaining new patterns. For instance, the pattern producing the numbers in Diagonal 3 becomes clearer when noticing that the difference between successive terms is 1, then 2, then 3, etc.

Students will also notice patterns in the horizontal rows in the triangle. For instance, they will see that each row has one more number than the row above it. They also can see that every other row has repeated numbers in the middle and notice the numbers match each other moving in opposite directions from the center.

It is harder for students to notice patterns that are not found in the diagonals or in horizontal rows, like the pattern where one number is the sum of the two numbers directly above in the previous row or the "hockey stick" pattern, and patterns that require thinking beyond the numbers represented in the triangle, like the pattern of row sums being powers of two. Not only is noticing these patterns more difficult, but even determining what to call these patterns so that others can follow what is being described is challenging. While students can name and/or point to examples of numbers that illustrate the pattern that was noticed, it helps to press them to name the arrangement so that others can appreciate that there is a general rule for how the pattern looks and search for new instances of the pattern. Calling a pattern the "Hockey Stick Pattern" may provide access whereas a description of the same thing can be much more opaque, such as, "go down a diagonal and then include a number in the opposite direction at the end." This is a good example of how a naming the arrangement may more clearly and succinctly convey the pattern that is being described.

One way to support students in expressing their understanding of how a pattern is functioning is to encourage them to extend the pattern. For instance, students can be asked to try generating the next row in the triangle or particular terms in the next row. They draw on their knowledge of the pattern to produce new numbers and can be asked to explain how they know a particular number "works" in a particular position in the triangle. The wonderful thing about the triangle is that there are multiple ways in which to generate and justify any new number. This feature supports discussions among students that are engaging and also mathematically substantive.

What mathematical practices are particularly relevant for working on this problem?

There are many mathematical practices that are used when noticing and describing the patterns in Pascal's Triangle. Four focal practices are identified below, and ways in which the engagement is supported by the problem are described.

MP3. Construct viable arguments and critique the reasoning of others.

> The heart of this task is describing the reasoning used in defining the patterns. In particular, identifying and naming patterns in Pascal's Triangle involves making conjectures by reasoning inductively and testing them out through analyzing different cases.

MP6. Attend to precision

> When engaging with the task, one must develop ways of describing which numbers are involved in the pattern. This often requires describing the pattern numerically and considering how the numbers are spatially arranged within the triangle. The language, which is sometimes invented, must be clear to others and as a result often requires adjustment in the precision of what is said.

MP7. Look for and make use of structure

> The triangle is built or structured according to specific rules or patterns. Articulating the rules that generate new numbers in the triangle reveals the structure of the triangle. Patterns are identified by looking for structure and frequently shifting perspective to notice new structures within the triangle.

MP8. Look for and express regularity in repeated reasoning

> Structure and repetition of reasoning are inherent in Pascal's Triangle. The triangle is formed by repeatedly adding two numbers in an upper row and writing the sum in the row beneath. Some patterns, such as "The Hockey Stick Pattern", are evidenced by repeated use of the pattern in various parts of the triangle.

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