*Reshaping Mathematics
for Understanding*

130 m

Measurement

160 m

50 m

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300 m

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Contents

Acknowledgments

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About the series . . .

Reshaping Mathematics for Understanding

Reshaping Mathematics for Understanding is a series of fourteen units suitable for sixth- through eighth-grade students that addresses important topics in middle-grades mathematics, including geometry, measurement, proportional reasoning, rational numbers, probability and statistics, and algebra. The entire series is designed to help students learn to think mathematically. It focuses on developing students' understanding of mathematical concepts and on their ability to draw connections among these concepts. It may serve either as the primary resource in the mathematics curriculum or as a complement to other material. The titles for the full series are listed below.

- Getting Started
- Motion Geometry
- Measurement
- Polygons
- Dilations
- Fractions
- Decimals
- Ratio and Proportion
- Area of Polygons
- Solids
- Probability and Statistics
- Integers
- Algebra Patterns and Relationships
- Number Theory

Access for Every Student

The problems and tasks in the *Reshaping Mathematics for Understanding* series are designed to enable every student to approach mathematics through familiar contexts. Building problem situations on students' past experience makes the study of mathematics more accessible and allows them to expand their thinking. Similarly, to promote genuine engagement, many problems have more than one solution path, giving students opportunities to choose the strategies they prefer using to solve problems. Additionally, to give students opportunities to interact with each concept at varying levels of abstractness and generality, the lessons present new ideas over several days. This design feature also allows students to learn at different rates.

Learning with Understanding

Students can learn only what they truly understand. To learn, they must understand the concepts that underlie operations; they must make connections among processes and concepts; and they must know when and how to apply concepts and operations to problems. Understanding mathematics means doing more than learning how to perform calculations to get correct answers. The topics in this series, developed through problems and lab explorations, encourage students to deepen their conceptual understanding through the practice of reasoning and problem solving.

Role of Visual/Spatial Thinking and Reasoning

Mathematics can and should be a lively course of study for students. It should engage them in active inquiry and give them many opportunities to explore problems whose solutions add to their understanding of the world. For many students, however, mathematics involves merely manipulating numbers. With this limited view, they do not learn how to use multiple contexts, tools, and strategies to solve problems or how to integrate mathematical concepts into broader contexts. Transformational (motion) geometry, which plays an important role in the sequence of the concepts in this series, emphasizes the use of visual contexts and spatial thinking. Lesson discussions further encourage students to use their understanding of spatial relations to make connections among concepts. The Motion Geometry and Dilations units, in particular, give students valuable experience with transformations and enhance the study of many related topics.

Unit Design

Each unit in the *Reshaping Mathematics for Understanding* series can be used individually, or units can be used in clusters. Throughout the series, references among units direct teachers and students to tasks that will help them connect their understanding of new concepts to related experiences.

The lessons in each unit are uniquely designed to enable students to progress through a sequence of tasks that maximize learning with understanding. Rather than present a topic in its entirety in one day, the lessons develop concepts over time. Most concepts begin with an open-ended problem that draws on students' previous experience and intuition and allows for multiple responses. The variety of solutions

students propose to such a problem helps the teacher assess what background knowledge students bring to the topic. The problems and tasks that follow develop the concept through a sequence of approaches that provide several direct examples of the concept, raise questions for clarification, offer alternative viewpoints, and prompt students to summarize ideas.

The Lessons

There are two forms of lessons in each unit, problem sets and in-class labs. Problem sets consisting of three to five problems are designed to cover several concepts related to the unit topic. Problem sets should be assigned for homework and discussed in class the next day. When students find a problem too difficult to solve on their own, they should be instructed to write questions to help them solve it and to ask the questions in the discussion. Students work on the labs in small groups in class and debrief afterward, giving them an opportunity to work collaboratively and to concentrate on one strand.

The class discussions are essential in helping students build conceptual understanding. Sharing their solutions and questions allows students to reflect on their thinking and to consider input from others. Both the teacher and students share the responsibility for making discussions productive. As they solve the problems and complete the lab tasks, students explain their thinking, offer alternative responses, and ask questions. Teachers facilitate, asking strategic questions to focus students' thinking on critical ideas. In guiding the discussion, teachers should ensure engagement in the learning process by encouraging students to monitor their learning and by providing a safe, open learning environment in which to share, discuss, and address misconceptions.

Assessment

Except for the Getting Started unit, designed to orient students to the series, all units include suggested assessment items. To check periodically that students understand and that they are participating in the class discussions, teachers can also create "instant" quizzes to give the day after a problem set discussion or lab debriefing. These quizzes should be unannounced and should take between five and ten minutes at the start of class. They have three purposes:

to emphasize the importance of student discussions;

to emphasize the value of information shared by students;

to highlight the importance of asking questions to clarify understanding.

In creating an instant quiz, teachers should focus on students' ideas from the discussion and write two or three short-answer questions as follows:

- a question that is content-based or refers to an understanding that has been established.

  Example: What did we decide it meant to measure the perimeter of a polygon?

- a question that refers to someone's method for solving a problem or alternative point of view.

  Example: What method did Leslie use to find the area of the trapezoid?

- a question that highlights an undefined assumption or a statement that needed clarification.

  Example: What did Jose mean by *proportional*?

Materials

A list of the materials needed for each lesson appears at the front of every unit. Although students need no special materials or equipment to complete the homework problems, it would be helpful to have them available during discussions. Encouraging students to use a broad range of tools to explore and express their thinking promotes greater understanding.

About the unit . . .

Using real-world and other visual/spatial contexts to study measurement is a powerful way to help students learn important mathematics principles. Measurement tasks and problems draw on concepts and skills from many areas of mathematics, such as number and operations, geometry, graphing, and proportional reasoning.

The Measurement unit gives students the experiences they need to understand the fundamental measurement concepts of unit and the iteration of unit. While students explore the properties of geometric measurement (length, area, volume, and angle measurement), the problems and activities in the unit help them understand procedures for measuring objects. They distinguish among methods for measuring different attributes of objects and analyze formulas for measuring the perimeter and area of rectangles and the volume of rectangular solids. Near the end of the unit they begin to explore the connection between measuring the area of a rectangle and the areas of other polygons.

Units of Measure

Measurement is a matter of comparing the size or quantity of an object with some fixed size or quantity, a *unit*. Students must understand this fundamental concept of a unit before they can truly understand measurement. A statement describing how long an object is, for example, is useful only when you know something about the unit used to measure it. Yet, in presenting measurement tasks, many texts require students to measure objects using already defined units. Although these tasks give students practice

measuring and reading a result, they do not require them to reason. Students who have not had the opportunity to reason about units have difficulty understanding ideas such as the inverse relationship between the size of the unit and the number of units needed to measure an attribute (as one increases, the other decreases).

Properties of Measurement

If a tall, slender dog and a short, husky dog walk down the street together, which one do we describe as *large*? When we talk about the size of objects, we must identify the attributes we are focusing on before the discussion can be meaningful. Students need a good sense of the properties of measuring those attributes. If they understand why linear measurement is one-dimensional, area measurement is two-dimensional, and volume measurement is three-dimensional, they will understand the procedures we use to measure those attributes and why we use the units we do. Additionally, they are less likely to confuse these attributes and will be more successful solving problems involving linear, area, and volume measurement.

Formulas

Using measurement formulas is an efficient way to count iterated units. While some students become very adept at remembering and applying formulas to problems, others struggle because they lack sufficient understanding of the underlying principles. Before using standard formulas, all middle-grades students must have the opportunity to explore a variety of methods to measure geometric objects. They need time to build an

understanding of why certain methods might be preferable to others.

The Measurement unit activities and problems promote discovery and reasoning about formulas. Students are expected to use reasoning to understand why formulas they may have already had experience with make sense. For example, although many students may have already learned the formula $A = l \times w$ for finding the area of a rectangle, few really understand it well. The problem and lab activities and accompanying discussions are designed to elicit mathematical reasons why that formula works.

Decomposition/ Recomposition

Decomposition/recomposition is a powerful way for students to explore and understand measurement formulas. It is a process that involves cutting up a shape into parts, acting on the parts, and then re-forming them into a whole shape. Several units (Measurement, Polygons, Area of Polygons, and Solids) present problems and activities using decomposition/recomposition strategies to derive area and volume formulas. This process is mathematically grounded in a key principle of transformation geometry: rigid motions preserve congruence (Motion Geometry unit). Thus, if students know the formula for finding the area of a rectangle, they can find the area of a parallelogram by moving its parts around to form a rectangle.
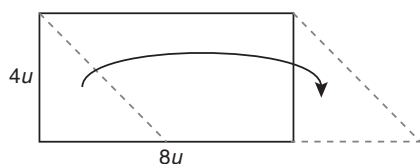
The parallelogram's area is equal to a rectangle with $b = 8u$ and $h = 4u$.

The areas of the two shapes will be the same because rigid motions were used to re-form the rectangle into a parallelogram.

The Strands in Measurement

Because of the strong connection between geometry and measurement, students' understanding of measurement may be greatly enhanced by a background in motion geometry. Thus, many concepts in the Measurement unit are developed through an understanding of transformations. See Motion Geometry unit.

Concept of Unit: Using nonstandard units before they use standard units encourages students to reason about the units. They cannot solve problems merely by reading an instrument. The problems in this strand require students to estimate measurements and relate the size of the unit to the number of units they need to measure a particular attribute. By counting whole and partial units, students practice and develop the skills needed to measure with reasonable accuracy. By emphasizing the connection to transformations, particularly translations and rotations, they develop an understanding of measurement as the iteration of a unit.

Parallel and Perpendicular Lines: The problems in this strand use motions to develop the concepts of parallel and perpendicular lines. Parallel lines are understood as translations or as 180° rotations, and perpendicular line segments as 90° rotation images of original line segments.

Angles: The concepts of an angle and angle measure are connected to the rotation motion. A ray and its rotation image form the sides of the angle, while the amount of rotation determines the angle measure. This approach helps students avoid the misunderstanding

that may result from thinking of an angle's measure as the amount of opening or distance between two rays. The problems in this strand do not include using a protractor; instead, they emphasize constructing and comparing angles using rotations.

Perimeter: The perimeter strand emphasizes the properties of linear measurement and the units used to measure length. Students focus on what types of situations require linear measurement, what skills they need to measure accurately and to read instruments correctly, and on how to derive formulas for measuring perimeter. They solve problems that help them observe how the perimeter and area of rectangles relate to each other.

Area: By stressing the properties of area measurement, several problems and activities in the area strand help students distinguish between measuring area and measuring length. Problems that draw attention to the type of units used to measure area help students understand why we use some form of the basic area formula *height* $\times$ *base* to measure the area of many plane shapes. At the same time, the strand includes problems that expose students' tendency to apply a simple $l \times w$ formula inappropriately.

Volume of Rectangular Solids: This strand highlights the three-dimensional nature of volume measurement. Students explore measuring capacity and discuss the properties of the unit used. To understand the formula for measuring the volume of a rectangular solid, they must understand volume as layers of area, each one unit high. In this way, students can see the connection between area measurement and volume.

Concept Development

The numbers in this matrix refer to the problem numbers in each lesson.

STRAND	PS 1	Lab A	Lab B	PS 2	Lab C	PS 3	PS 4	Lab D	PS 5	PS 6	Lab E	PS 7	Lab F	PS 8	Lab G	PS 9	Lab H	PS 10	PS11
Concept of unit																			
Size	1																		
Iteration of units	3																		
Standard/nonstandard			1-7							3									
Units of length		all		4															
Referent whole				3			4		2										
Characteristics of linear measurement						3													
Parallel & Perpendicular Lines																			
Relationship to translations	2					1													
Relationship to rotations							2												
Relationship to motions												2							
Linear measurement																			
Iterating units						5													
Angles																			
Angle as a rotation				1		4	1							1					
Constructing an angle									1	1									
Measuring angles									1, 3										
Perimeter																			
Definition of perimeter				2															
Characteristics of linear measurement					1														
Measuring perimeter					2, 3	2	3						4						
Area																			
Characteristics of area measurement								all	2			1							
Conservation of area									4		all					1			
Measuring area									4		all	3	all	2, 3		1, 4		1,2	
Parallelograms																			1
Volume of Rectangular Solids																			
Characteristics of volume measurement														4	all	2			
Measuring volume																3	all	3	3
Volume layer																		3	2

Materials

Materials used throughout Measurement

Straight-edge,* 1 per student
Centimeter ruler, 1 per student
Tracing paper,** 2–3 sheets per student per lesson
String, 1 ball of twine per class
Scissors, 1 per group
Linking cubes, 36 per student

Special materials for Measurement

Lab A   Long sheets of paper like chart or butcher paper, 1 sheet (approximately 12 inches by 60 inches) per pair
Colored pen, 1 per student

Lab C   Paper clip, 1 per student
Orange Cuisenaire® rod, 1 per student

Problem Set 4   Lined paper, 1 sheet per student

Lab D   Flat, traceable object, 1 object per student
Square tile, 1 per student
Half-sheets of paper, 2 per student

Lab E   Square tile, 17 per student

Problem Set 7   A Geoboard

Lab F   Grid paper with 1 cm squares (see appendix), 1 per student

Lab G   Grid paper with 1 cm squares (see appendix), 1 per person
Popcorn, approximately 1 quart per group

Problem Set 11   A set of twelve identical textbooks
*Straight-edges without unit markings are preferable to rulers. Blank index cards could also be used.
**Patty paper cut into 5-inch squares is an economical alternative to the art paper used for tracing.

From Motions to Measurement

Problem Set 1, No. 1

Strand: Concept of Unit

Focus: Size

Task: Using unit sense and properties to compare the sizes of drawings of real-life objects

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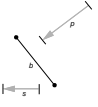
PROBLEM
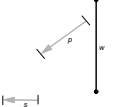
1. a. Order the figures below by size.

   b. Explain your method for ordering the figures.

ANSWER

Answers will depend on the attribute students choose to compare the figures. The discussion should include the following answer:

1. a. B, D, C, and A

   b. The figures were ordered by their areas, or by how much of the paper the drawing covered.
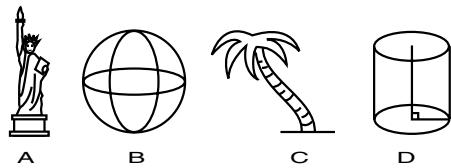
Teacher's Insight

The problem direction is deliberately open-ended. Students may compare the figures based on the size of the pictures or on their actual size in real life. For example, students may say the actual Statue of Liberty is larger than a ball, a tree, and a container. The problem also allows students to identify which attributes one may use to talk about size. The discussion should clarify the importance of understanding the basis they use to make comparisons.

Discussion

Have students share and discuss a variety of answers.

How did you order the figures?

What did you look at to make your comparisons?

Why might someone order them differently?

Why would we want to agree on a method to compare sizes?

Problem Set 1, No. 2

Strand: Parallel and Perpendicular Lines

Focus: Relationship to translations

Task: Identifying parallelism by translating lines

PROBLEM

2. a. Translate line segment *b* using arrow *p*.

    b. Translate line segment *b* using arrow *s*.
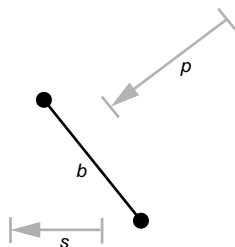
    c. Translate line segment *w* using arrow *p*.

    d. Translate line segment *w* using arrow *s*.
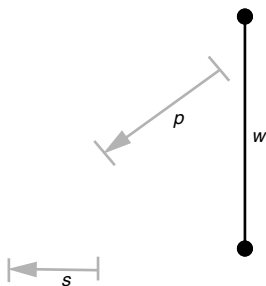
    e. On each translation, what do you notice about the original position of the line segment compared to its position after you translated it?

Teacher's Insight

A translation, or slide, is a transformation in which a shape slides without turning. Every point moves the same distance in the same direction. When a line or line segment is translated, it and its image create parallel lines. Students who have not had experience with transformations may need introductory activities before they do this problem. For more problems to help students understand transformations, see the Motion Geometry unit in this series.

ANSWER

2. a–b.
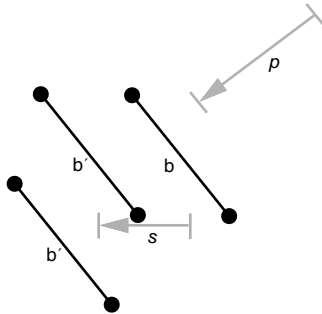
c–d.
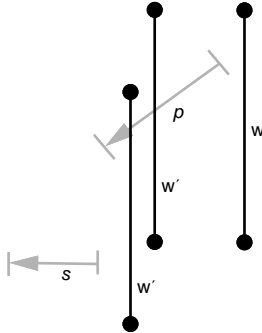
e. Multiple solutions are possible, including the following: The original line segment and its image are parallel.

Discussion

What do you notice about the original and each image of the line segment?

How could you check to be certain two lines are parallel?

Draw another translation arrow that would result in an image parallel to its original line segment.

What makes two lines or line segments parallel?

Problem Set 1, No. 3

Strand: Concept of Unit

Focus: Iteration of units

Task: Iterating a unit to create a segment of a desired length

PROBLEM

3. For homework Malia had to draw a line segment that was the length of her foot from the tip of the big toe to the heel. Her only tools were a pencil, a piece of paper, and a broken piece of a straight-edge like the one pictured below. Describe how Malia could accurately draw the line segment. Draw a diagram to show what she should do.
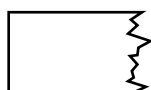
Actual size

ANSWER

Multiple solutions are possible, including the following:

3. Malia could place her foot on the paper so that her heel aligns with the bottom of the paper, then trace her toes on the paper. Malia could then fold and crease the paper the length of her foot. Finally she could use the straight-edge to draw a line segment in the fold from her heel to the tip of her longest toe. She will need to translate the straight-edge to draw the entire segment.

fold

fold

trace
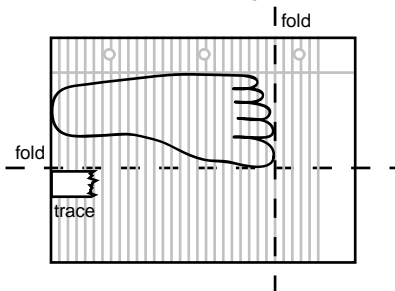
★HOMEWORK

None

Teacher's Insight

(This problem may need to be previewed to help students understand the task before working on it as homework.) The problem poses two dilemmas for students. First, they must determine how long the unit is. Since the end of the broken piece is jagged, students have to decide where to even it off. Next they must decide how to move a small unit physically to measure the length of a foot. Repeating (or iterating) a unit is the basis for linear measurement. They should be able to describe the length of their feet in terms of the number of iterations needed to draw the segment.

Discussion

Check for understanding of the problem situation.

Have students share and discuss a variety of answers.

What did you do to solve the problem? (Encourage students to generate lots of ideas for solutions.)

What problems do you think Malia had with this broken straight-edge? (If students talk about the jagged edge, use this opportunity to explore how to iterate units.)

How could Malia be sure her line segment is the right length without making any markings on the straight-edge piece?

Would everyone's foot need the same number of tracings? Why? Why not?

Compare this exercise to a geometric motion.

Lab A

Measuring with "Me" Units

Lab A, Part I

Strand: Concept of Unit

Focus: Units of length

Task: Practicing and discussing procedures for measuring length that result in accurate measurements

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PROBLEM

MEASURING WITH ME

You will use each body measure to measure across your classroom.

Fill in the chart. Be sure to estimate before you measure.

	MY ESTIMATE	MY MEASURE
Palm		
Arm span		
Baby steps		
Paces		

Draw and label your own body measure.

1. How did you make your estimates?

2. Explain your method for measuring. Include where you took the measurement, how you made the measurement, and how you kept track of the measurement.

3. What problems did you and your partner encounter while you were measuring?

4. How did you solve those problems?

5. What motion might relate to your measuring?

Teacher's Insight

Linear measurement is one-dimensional. It uses the iteration, or repeated placement, of a unit of length along a given path. In part I of this lab students use parts of their bodies as units of length to measure across the classroom. They work in pairs to estimate the number of units and then take the measurements. Estimating before measuring helps them develop number sense with regard to a unit's size. Partners must establish procedures to make their measurements as accurate as possible. Some of the challenges students face when they do this lab are where to make the measurement, how to measure in a straight line with objects in the way, and how to place units and keep track of the unit count. The discussion that follows the activity should highlight the procedures students followed. Students should clarify the connection between their procedures for measuring and the nature of linear measurement.

ANSWER

Multiple solutions are possible, including the following:

1. Visually compare the length of one unit with the length of the classroom.

2. Measure along a wall because it is a straight path. Assign one partner to indicate where to place each unit and keep count while the other partner positions the body measures. Students should describe the iteration of a unit and the end-to-end placement of the unit to measure the distance. They should also develop methods for estimating fractions of units.

3. Common problems include physical obstacles that can't be moved and other pairs of students measuring along the same path.

4. Students may estimate the distances around the physical obstacles by comparing the obstructed distance with one they have already measured, and they learn to work with others by taking turns to measure the path.

5. Measuring linear paths relates to translations in motion geometry.

Discussion

How did you decide where you would measure the distance across the room?

Describe your procedure for measuring the length of the room.

Were some body parts easier than others to use as measuring units? What made them easier to use?

How did you decide to create your own body measure?

How did you count your units? How did you keep track of the count?

How accurate do you think your measurements were?

What would have helped you be more accurate?

Lab A, Part II

Strand: Concept of Unit

Focus: Units of length

Task: Creating a measuring tape calibrated in body-part units

PROBLEM

Part II

1. Choose one of the body measures from part I to use as a measuring unit. What is the name of your unit?

   How long was the classroom using your chosen body measure?

2. Make a measuring tape using your body measure as the unit. Describe how you made it.

3. Measure across the classroom using your tape. Be sure to measure the same length of the classroom as you did in part I. How long is the classroom as measured with the tape?

4. Compare the measurements you got in part I and II. Are they the same or different? Explain why they may have turned out that way.

Teacher's Insight

Partners choose one person's body part to create a tape measure. Their drawings of units on the tape should illustrate the procedure they used to measure the length of the room. For example, some drawings show how students placed the body-part unit with no overlaps and no spaces between the units. The drawings should show how students iterated the unit and how they kept count of the number of units. Students who choose larger body-part units may have to estimate a fractional part of a unit if it does not fit the length of the tape exactly.
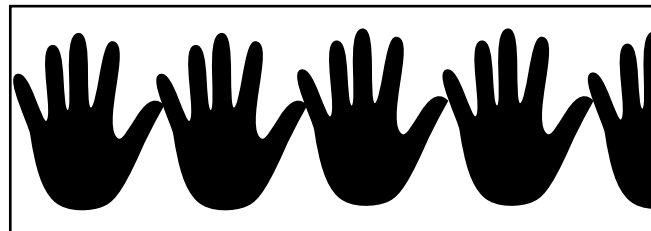
ANSWER

1–4. Answers will vary with the choices students make. Measurements from parts I and II should be very close. The tape measure is easier to use and gives a more accurate measurement than the actual body part because the tape measure unit remains constant. Folding the tape is a useful technique to determine fractional parts.

★HOMEWORK

None

DISCUSSION

How did you use your body-part unit to make the measuring tape?

Why did you choose the unit you used?

How did you number the units?

How did you count partial units? How did you number partial units?

If students used fractions, ask them how they decided what fraction the partial unit was.

Compare your measurement with the body part in part I with your measurement using the tape.

Were your procedures the same or different? Explain.

Did you get the same measurement with both methods?

Would you expect to get the same measurement in part I as in part II? Explain why or why not.