Reshaping Mathematics for Understanding

Motion Geometry

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Contents

About the unit . . .

The Motion Geometry unit introduces students to concepts of geometry they will use throughout middle-grade and higher-level mathematics courses. These concepts, presented through the study of transformations, provide a framework for other important topics such as number, measurement, proportional reasoning, and graphing on the coordinate plane. As students learn about reflections, translations, and rotations (flips, slides, and turns), they are also learning about the properties of the motions and how they affect objects. The problems in this unit require students to manipulate drawings physically, to be accurate in their work, and to use precise language in analyzing the results of the motions.

Geometric Thinking

Motion geometry in the middle grades can provide a powerful context for students to develop their reasoning processes. Problems are carefully designed to help them progress from using the informal visual and descriptive bases for thinking about geometric objects which are suitable for elementary grades, to using more formal reasoning based on the logical arguments and justifications that are required in advanced courses.

Motion is a part of everyone's daily life—riding a Ferris wheel, sliding into home plate, or flipping a coin. The study of motions—reflections, translations, and rotations—allows students to draw on their experience and their intuition as they strive to understand the physical world. For example, they can compare two triangles by flipping one over to match the angles, sliding one on top of the other to check the areas, or turning one around to compare its shape to another's. These "common sense" methods, which are grounded in mathematical principles, help students to reason and to justify the relationships between the triangles.

The Language of Motion Geometry

Although many of the ideas from motion geometry relate to familiar experiences, we use special terminology to name and describe transformations. Some of the terminology can be confusing to students. For example, the term translation means something different in a foreign language class than it does in a mathematics class. The problems in this unit help students build on their previous conceptions to learn new mathematics terminology, and the suggested discussion questions encourage them to clarify their understanding and to re-define terms in a mathematics context.

Recording Motions on Paper

Solving problems in motion geometry requires students to manipulate drawings physically using pencil, paper, and other tools. They must trace given figures with a reasonable degree of accuracy, use lines and arrows in particular ways, and label drawings effectively to communicate to others how they solved a problem. Some middle-grades students have difficulty achieving this level of precision. The teacher and students together should determine appropriate expectations for neatness. However, it is important that all students cultivate habits of mind that promote care and mathematical correctness. At the same time, class discussions should include opportunities for students to share and practice techniques for making drawings accurate, and should highlight how the techniques and procedures reflect the mathematical properties of the motions. In this way, the study of the physical and the conceptual components of transformations complement each other. (See page 107 for suggested drawing methods.)

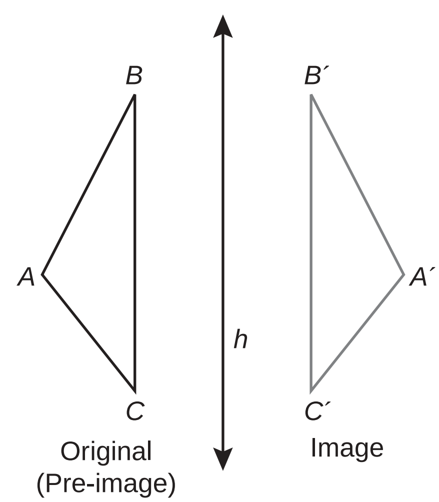
The Strands in Motion Geometry

Introduction to Motion: Students begin by using real objects to explore motion. They experiment with different types of motions, informally describing the properties of the motions they perform. They compare an object before and after the motion; for example, they may focus on its orientation. As they observe that the size, shape, and other physical properties of the objects do not change with rigid motions, students develop a sense of congruence.

Representation: Because an actual motion cannot be preserved on paper, we must use representation to convey how an object should be or has been moved. The early tasks in this unit give students opportunities to invent ways to represent the motions they perform and to explain their drawings. These experiences help students understand the importance of using a common scheme to represent motions and to make sense of some conventional schemes.

The problems in this unit use a set of "tools" to represent how one should move an object or to convey how an object has been moved. These tools are the line of reflection, the translation arrow, and the rotation arrow and center of rotation. Each is discussed in its related strand section.

Reflection: Most students have had some experience with reflections, which is the easiest motion to perform. The sequence of problems in the reflection strand requires students to reflect figures, find lines of reflection, and locate the position of objects before they were reflected. Reflections are performed over a line of reflection. Students may devise different techniques for doing the reflection, but they must follow all conditions as presented in the problem. For example, they cannot move a given line of reflection to a more convenient location on their paper. Students should be able to draw with accuracy both figures and the line used to create the reflection.



The original triangle (in black) is reflected *over* line *h* to get the image (in gray).

Students may see the connection between their experience of reflections and solving problems involving line symmetry, graphing parabolas, or classifying quadrilaterals.

Translation: Translations move a figure a fixed distance in a given direction. Translation arrows (vectors) represent the distance and direction for moving the figure. The problems in this strand give students experience in translating figures, locating figures before they were translated, and suggesting arrows that might have been used in a particular translation. Students need to practice using the translation arrow "tool," which helps them make their drawings accurate. They are expected to draw translation images and arrows accurately and to label the figures correctly.

Students who understand translations can apply this experience to measurement topics—length, area, and volume—to parallel lines, and to graphing on the coordinate plane.

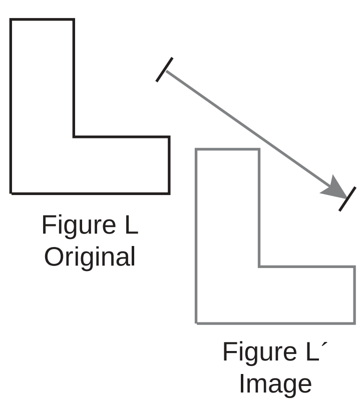


Figure L and its translated image.

Rotation: A rotation is a transformation that turns a figure around one point (or, to use the mathematical term, "*rotates* a figure *about* a point") called the center of rotation. Students learn about the tools used to rotate objects and practice doing rotations. They use degrees and fractions to describe rotations.

Rotations are the most challenging motion for students to do. They may not have had enough experience with circles or be sufficiently adept at using a compass to make precise drawings for problems, that, for example, ask them to locate a center of rotation. They should, however, be able to approximate where a center of rotation is and to sketch possible rotation arrows.

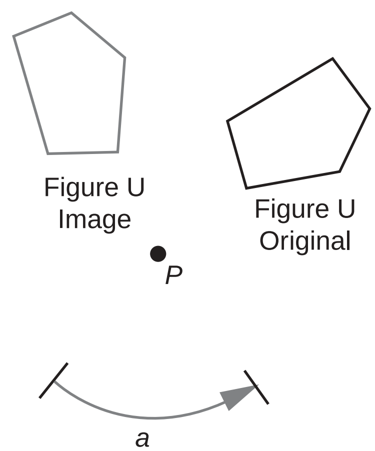


Figure U and its rotation image using arrow *a*.

Rotations help students understand angle measurement, symmetry, and slope.

Concept Development

STRAND and FOCUS	Lab A	PS 1	Lab B	PS 2	Lab C	PS 3	PS 4	Lab D	PS 5	PS 6	PS 7	PS 8	PS 9	PS 10
Introduction to Motion														
Moving Objects	1–5	2												
Orientation	6	2, 3, 4												
Congruence	1–4, 6													
Representation														
Representing Motion	5													
Reflection														
Properties of a Reflection		1	1											
Line of Reflection			1, 2	1, 2, 3	4	1, 4	1	4	1		2		2	
Reflection Image										1				
Comparing Motions												1		
Finding the Line of Reflection														1
Translation														
Properties of a Translation				4	1, 2	2								
Direction of Motion					3									
Distance of a Translation							4							
Translation Arrow							4	2	3	2, 3	1			
Translation Image												4	1	
Comparing Motions												1		
Multiple Translations													4	
Translation of a Translation														2
Rotation														
Properties of a Rotation						3	2, 3	1		4	4	2		
Center of Rotation								1						
Rotation Arrow									2, 4	4		3		
Magnitude of a Rotation								3			3		3	
Comparing Motions														
Arc Measure of a Rotation Arrow														3

Materials

Materials used throughout Motion Geometry

Geometric compass, 2 per group

MIRA®, 1 per student

Straight-edge*, 1 per student

Tracing paper**, 2–3 sheets per student per lesson

Special materials for Motion Geometry

Lab A

A variety of flat, traceable objects, 1 object per student

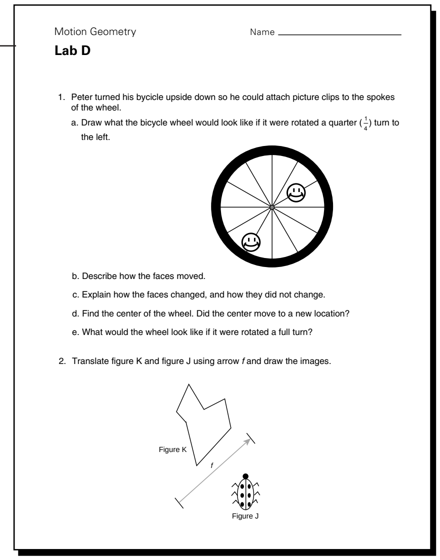
Blank half sheets of paper, 3 per person

*Straight-edges without unit markings are preferable to rulers. Blank index cards could also be used.

**Patty paper cut into 5-inch squares is an economical alternative to the art paper used for tracing.

Lab D

Developing a Method for Rotations



Motion Geometry Name

Lab D

1. Peter turned his bicycle upside down so he could attach picture clips to the spokes of the wheel.
   a. Draw what the bicycle wheel would look like if it were rotated a quarter ($\frac{1}{4}$) turn to the left.

   b. Describe how the faces moved.
   c. Explain how the faces changed, and how they did not change.
   d. Find the center of the wheel. Did the center move to a new location?
   e. What would the wheel look like if it were rotated a full turn?

2. Translate figure K and figure J using arrow *f* and draw the images.

Lab D, No. 1

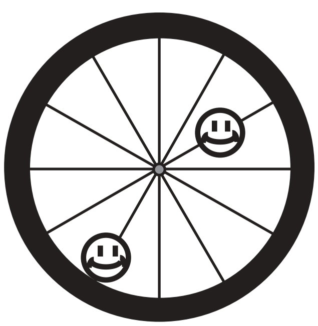
Strand: Rotation

Focus: Properties of a rotation and center of rotation

Task: Relating the movement of a familiar object to the geometric motion of rotation and relating the center of the wheel to the center of rotation

PROBLEM

1. Peter turned his bicycle upside down so he could attach picture clips to the spokes of the wheel.
   a. Draw what the bicycle wheel would look like if it were rotated a quarter ($\frac{1}{4}$) turn to the left.



   b. Describe how the faces moved.
   c. Explain how the faces changed, and how they did not change.
   d. Find the center of the wheel. Did the center move to a new location?
   e. What would the wheel look like if it were rotated a full turn?

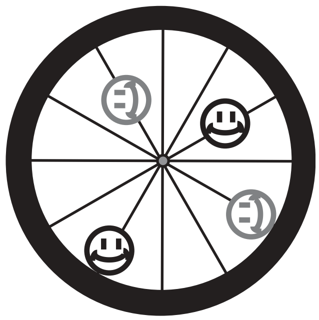
Teacher's Insight

In this problem students investigate the properties of rotation and use terms to describe a particular rotation. They need to clarify whether "turn left" means the top or the bottom of the wheel moves left. The image will depend on what they decide. Later problems will demonstrate the use or function of a rotation arrow to identify a starting point.

Students may think the faces "change" if they picture them sliding along the spoke. Have them focus on the size, shape, and features of the faces when deciding whether they change. The center does not move or change in any way.

ANSWER

1. a. If “rotate to the left” means the wheel rolls to the left, then these are the answers:



   b. The faces are turned on their side after the quarter turn.

   c. The location of the face changed. Each one looks as if it was moved three wires over. The faces are on the same wire of the wheel, and they are the same shape and size.

   d. No, the center holds the wheel to the bike so it stays stationary.

   e. The wheel would look the same as it did in its starting position.

Discussion

How did you decide how much a quarter turn was?

How did you decide in what direction to turn the wheel?

How did your decision about which way to turn the wheel affect the image?

How did you find the center of the wheel?

How did you decide what a full turn was?

How can we change the problem so there is no confusion about the direction to turn the wheel?

Lab D, No. 2

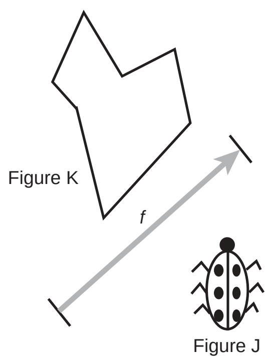
Strand: Translation

Focus: Translation arrow

Task: Demonstrating the function of a translation arrow

PROBLEM

2. Translate figure K and figure J using arrow *f* and draw the images.

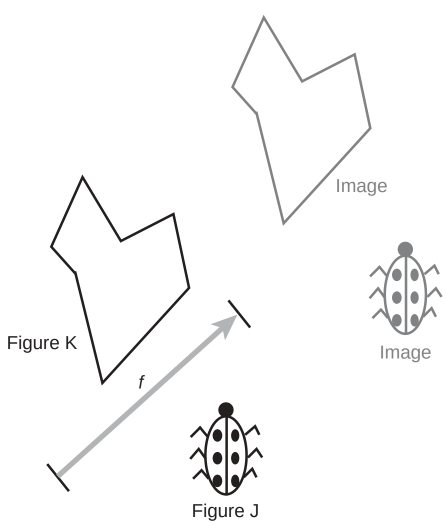


Teacher's Insight

This problem gives students experience with translating a figure using a specified translation arrow. Have them demonstrate their methods for doing the translation. Review the information the arrow gives them about the motion. Students sometimes translate each figure separately. If someone translates both figures at the same time, ask whether the result would be the same and how efficient translating both figures in the same motion might be. If no one translates both figures at the same time, ask students if it is possible and then have them do it to see that the outcome is the same.

ANSWER

2.



Discussion

Have students demonstrate the motion.

If needed, have students practice doing the translation.

How did you use the arrow?

How did you know where to start translating?

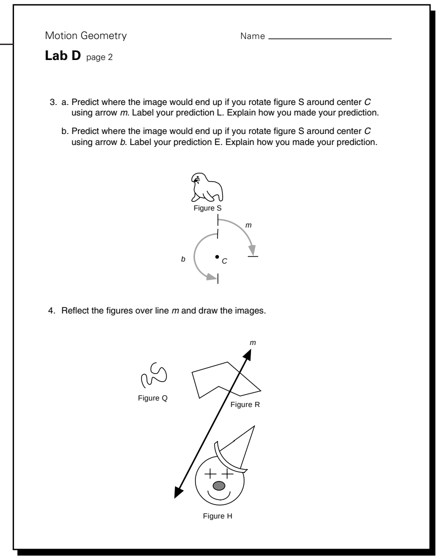
How did you know when to stop translating the figures?

How far did the bug move?

How far did figure K move?

What information do you get from the translation arrow?

How does that information help you do the translation?


Motion Geometry Name ____________

Lab D page 2

3. a. Predict where the image would end up if you rotate figure S around center *C* using arrow *m*. Label your prediction L. Explain how you made your prediction.

   b. Predict where the image would end up if you rotate figure S around center *C* using arrow *b*. Label your prediction E. Explain how you made your prediction.

4. Reflect the figures over line *m* and draw the images.

Lab D, No. 3

Strand: Rotation

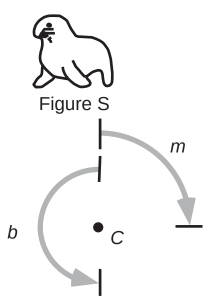
Focus: Magnitude of a rotation

Task: Predicting the location of a rotation image

PROBLEM

3. a. Predict where the image will end up if you rotate figure S around center *C* using arrow *m*. Label your prediction L. Explain how you made your prediction.

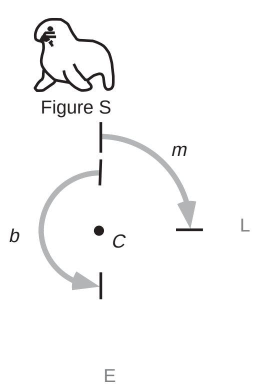
   b. Predict where the image will end up if you rotate figure S around center *C* using arrow *b*. Label your prediction E. Explain how you made your prediction.



ANSWER

3. a. The arrow is a quarter turn, so figure S should rotate that much. If the starting position is thought of as 12 o'clock, then a quarter turn is 3 o'clock.

   b. The arrow is a half turn going counterclockwise, so figure S should rotate from the 12 o'clock back to the 6 o'clock position.



Teacher's Insight

Students use their motion sense to predict the outcomes of rotations. Arrows *m* and *b* represent a quarter turn and a half turn, respectively. (Most students are familiar with these portions of a circle.) They may describe the turn in fractions or degrees. Whichever way they choose to express the amount of rotation, ask how they knew that was the amount. Students can test their predictions by performing the rotation during the discussion.

Discussion

How did you make your predictions?

How did you decide how far the figures would rotate?

Did you use the arrows? If so, how?

If students used fractions to make their predictions, ask how they knew arrow *m* was a quarter turn and arrow *b* was a half turn.

How did using the fractions help you?

How could you test your predictions?

Lab D, No. 4

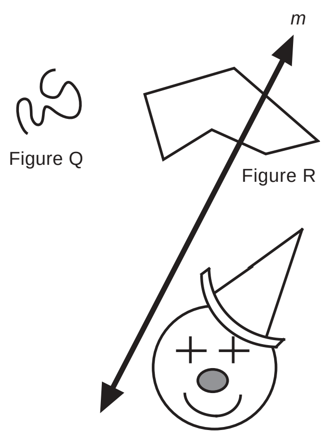
Strand: Reflection

Focus: Line of reflection

Task: Reflecting figures on both sides of the line of reflection, and a figure that is intersected by the line of reflection

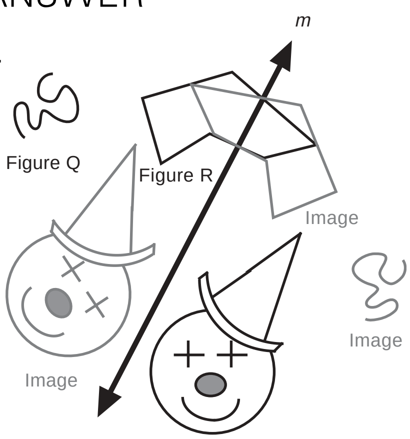
PROBLEM

4. Reflect the figures over line *m* and draw the images.



ANSWER

4.



★HOMEWORK

Problem Set 5

Teacher's Insight

Students may have difficulty reflecting the figure lying directly on the line of reflection. Check to see that they complete the reflection on both sides of the line.

Discussion

Have students demonstrate and describe their methods for reflecting the figures.

Is anything different about the drawing in this problem?

Did you use the same method to reflect all the figures?

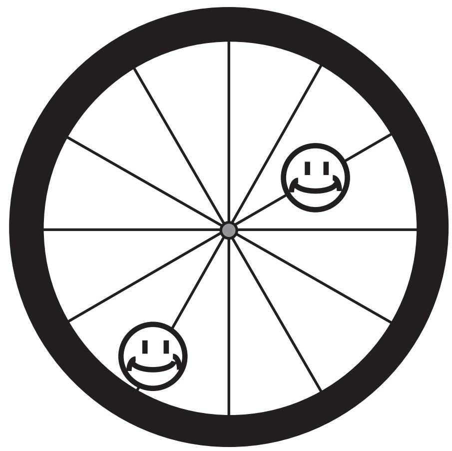
What do you notice about the location of each figure's image in relation to the line of reflection?

 Name ____________________

Lab D

1. Peter turned his bycicle upside down so he could attach picture clips to the spokes of the wheel.

   a. Draw what the bicycle wheel would look like if it were rotated a quarter ($\frac{1}{4}$) turn to the left.



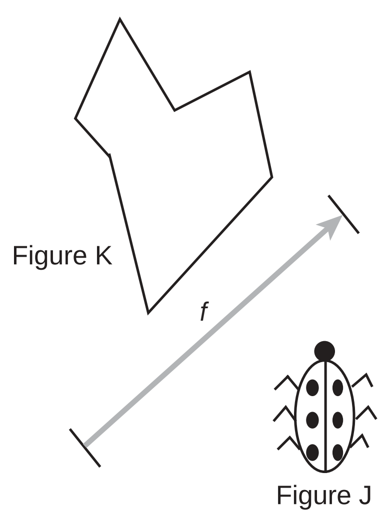
   b. Describe how the faces moved.

   c. Explain how the faces changed, and how they did not change.

   d. Find the center of the wheel. Did the center move to a new location?

   e. What would the wheel look like if it were rotated a full turn?

2. Translate figure K and figure J using arrow *f* and draw the images.

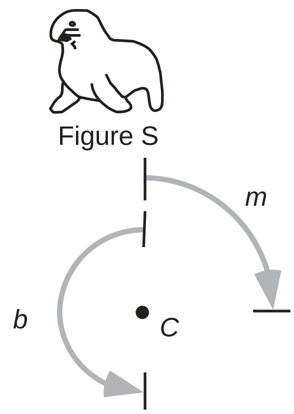


Name ______________________

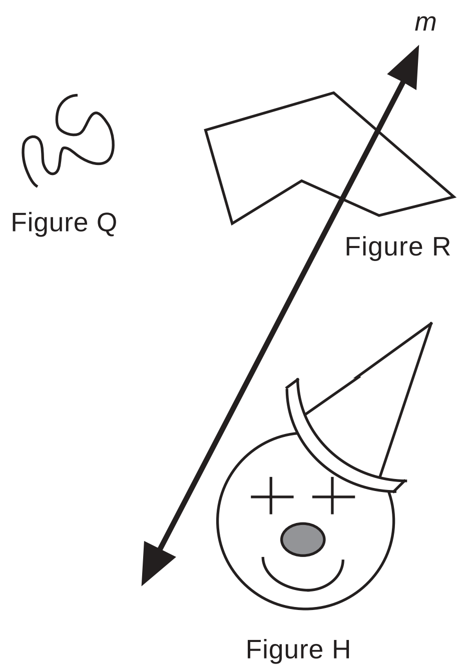
Lab D page 2

3. a. Predict where the image would end up if you rotate figure S around center *C* using arrow *m*. Label your prediction L. Explain how you made your prediction.

   b. Predict where the image would end up if you rotate figure S around center *C* using arrow *b*. Label your prediction E. Explain how you made your prediction.

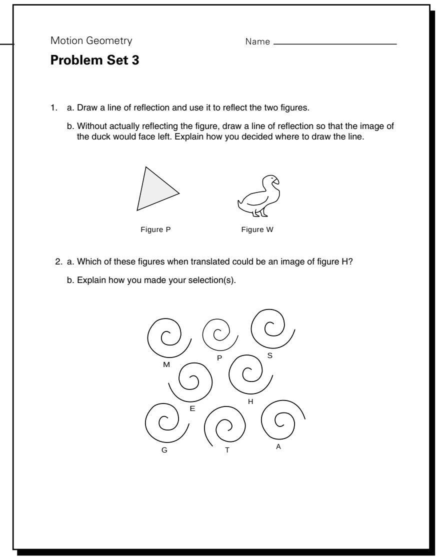


4. Reflect the figures over line *m* and draw the images.



Problem Set 3

Developing a Method for Translations


Motion Geometry Name ______________

Problem Set 3

1. a. Draw a line of reflection and use it to reflect the two figures.

   b. Without actually reflecting the figure, draw a line of reflection so that the image of the duck would face left. Explain how you decided where to draw the line.

Figure P Figure W

2. a. Which of these figures when translated could be an image of figure H?

   b. Explain how you made your selection(s).

Problem Set 3, No. 1

Strand: Reflection

Focus: Line of reflection

Task: Selecting lines of reflection and judging the effects of the placement of the lines on the images

PROBLEM

1. a. Draw a line of reflection and use it to reflect the two figures.

   b. Without actually reflecting the figure, draw a line of reflection so that the image of the duck would face left. Explain how you decided where to draw the line.

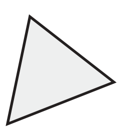


Figure P

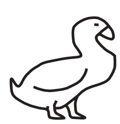


Figure W

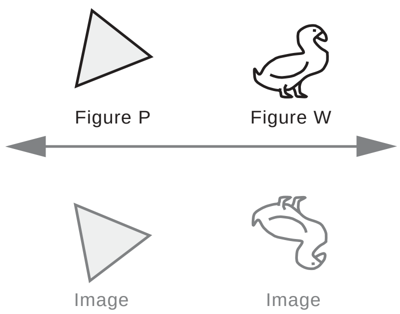
Teacher’s Insight

In the first part of the problem students can draw the line of reflection wherever they wish. Some may reflect each figure separately. Point out that the problem gives them the two figures to reflect; reflecting them separately changes the given problem. To help students understand, you could point out that reflecting the figures separately is like changing the problem 25 + 17 to 2 + 1 and 5 + 7.

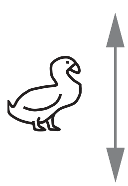
In 1.b. students use their motion sense and their knowledge about the properties of reflections to predict a reflection that will give the desired results.

ANSWER

1. a. Multiple answers are possible, including the following:



b. Multiple answers are possible, including the following:



Discussion

How did you decide where to draw your line of reflection in 1.a.? Is your answer unique? How many different ways could you solve 1.a.?

How did you decide where to draw the line of reflection in 1.b.? Is your answer unique? How many different ways could you solve 1.b.?

How does the line help you reflect the figures?

Explain how the location of the line of reflection affects the image.

Problem Set 3, No. 2

Strand: Translation

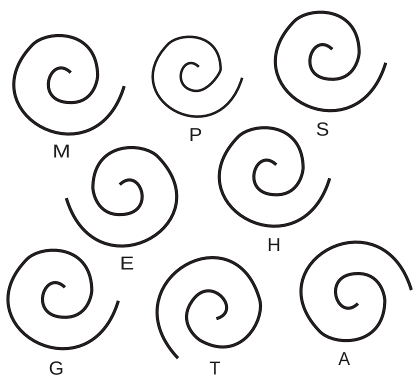
Focus: Properties of a translation

Task: Identifying and describing properties of translation images

PROBLEM

2. a. Which of the figures when translated could be an image of figure H?

   b. Explain how you made your selection(s).



ANSWER

2. a. S, M, and G.

   b. These images look like straight slides because they are facing the same way as figure H. This could be checked by tracing figure H and sliding it in a straight line to see if it will match up with the others.

Teacher's Insight

By contrasting examples and non-examples of translation images, students strengthen their understanding of the motion's properties. In the discussion ask students to explain the reasoning they used to choose the examples of translation images and the reasons the other figures are not examples. Also address the methods students used to test their perceptions. Some students will base their decisions only on observation. This is a good beginning, but they should also know how to use tests that are more mathematically certain. Help them identify a more robust strategy for testing their first guesses. One such test would be to trace a figure and translate it to see if it maps on to figure H. If students talk about the possibility that figure P is an image of a translation moving backward, point out that the paper is a two-dimensional context. The motion they are describing calls for a three-dimensional context.

Discussion

How did you decide which were translation images of figure H?

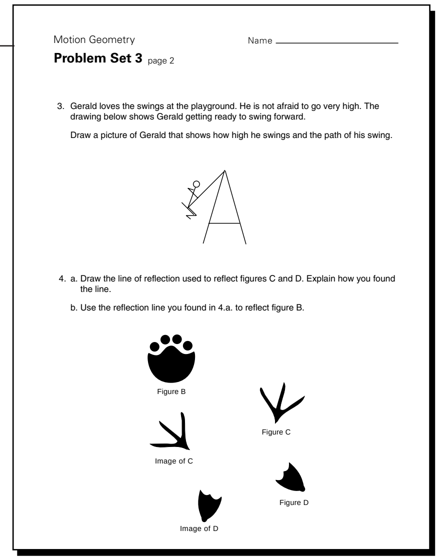
What did you notice about the figures you chose as translation images of figure H?

How did you test the images to see if they really were translation images?

How did you decide that the other figures were not translation images of figure H?

Why isn't figure T a translation image of figure H? What about figure P?

What is an image?


Motion Geometry
Name ____________
Problem Set 3 page 2

3. Gerald loves the swings at the playground. He is not afraid to go very high. The drawing below shows Gerald getting ready to swing forward.

   Draw a picture of Gerald that shows how high he swings and the path of his swing.

4. a. Draw the line of reflection used to reflect figures C and D. Explain how you found the line.

   b. Use the reflection line you found in 4.a. to reflect figure B.

Problem Set 3, No. 3

Strand: Rotation

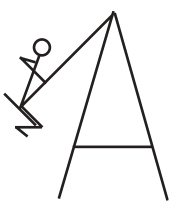
Focus: Properties of a rotation

Task: Identifying the general motion of a rotation around a center

PROBLEM

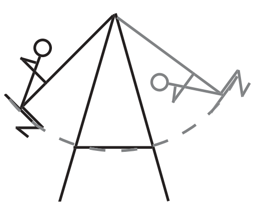
3. Gerald loves the swings at the playground. He's not afraid to go very high. The drawing below shows Gerald getting ready to swing forward.

   Draw a picture of Gerald that shows how high he swings and the path of his swing.



ANSWER

3. Multiple answers are possible, including the following:



Teacher's Insight

This problem introduces the rotation strand by focusing on the properties of a rotation, such as the shape of the path. Students discuss details of the properties in subsequent problems. In a rotation the original figure moves in a circular path around a fixed center of rotation. A rotation image is congruent to the original figure.

Students first encounter a rotation in the context of a swing. The swing moves in an arc whose center of rotation is where the swing's chains attach to the frame. The problem's open-ended nature elicits students' prior experience and lets you assess what they already know about rotations. Students usually use a curved arrow to represent the path of the rotation.

Discussion

How did you *decide* how high Gerald swung?

How did you *show* how high he swung?

In what direction did he swing? How could you show it?

How might you describe this movement?

What shape did Gerald's path resemble?

Problem Set 3, No. 4

Strand: Reflection

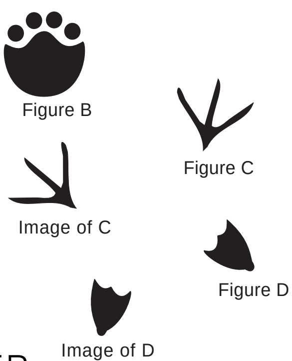
Focus: Line of reflection

Task: Identifying the line of reflection and using it to create a reflection image

PROBLEM

4. a. Draw the line of reflection used to reflect figures C and D. Explain how you found the line.

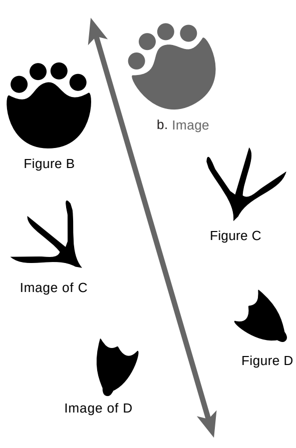
   b. Use the reflection line you found in 4.a. to reflect figure B.



ANSWER

4. a. The line of reflection is found by folding the paper so that the figures map over their corresponding images.

   b.



★ HOMEWORK

Problem Set 4

Teacher's Insight

Students use reversibility to find the line of reflection and then use the same line to reflect a given figure. In this drawing two original figures are on one side of the line of reflection and the third original figure is on the left. Help students focus on the details of the given drawing, especially the labeling. They should remember to label the drawing that represents their solution carefully.

Discussion

How did you know where to draw the line of reflection?

Is your answer unique?

Make as many observations as you can about the original and the image of a reflection.

Name ____________________

Problem Set 3

1. a. Draw a line of reflection and use it to reflect the two figures.

   b. Without actually reflecting the figure, draw a line of reflection so that the image of the duck would face left. Explain how you decided where to draw the line.

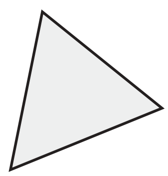


Figure P

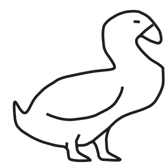
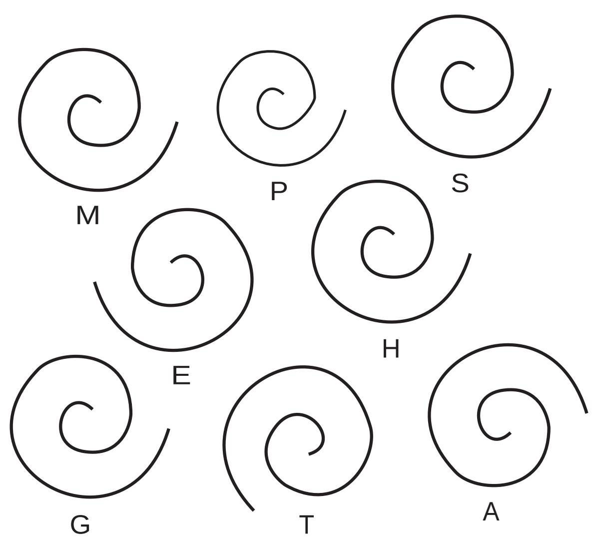


Figure W

2. a. Which of these figures when translated could be an image of figure H?

   b. Explain how you made your selection(s).

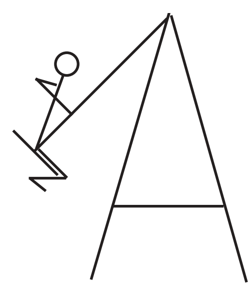


 Name ______________________

Problem Set 3 page 2

3. Gerald loves the swings at the playground. He is not afraid to go very high. The drawing below shows Gerald getting ready to swing forward.

   Draw a picture of Gerald that shows how high he swings and the path of his swing.



4. a. Draw the line of reflection used to reflect figures C and D. Explain how you found the line.

   b. Use the reflection line you found in 4.a. to reflect figure B.