Measure Up for Understanding

Teacher. Justin, what do you notice?
Justin. I notice that the bigger the unit, the lesser times you use it.
Wendy. And if the unit is smaller, the more times you have to use it.
Stephanie. It all depends on the unit.

This conversation between teacher and students took place during the fall semester 2003 in a first-grade class as part of Measure Up, an elementary mathematics research project at the Curriculum Research and Development Group of the University of Hawaii. In Measure Up, all students in grades 1–5 develop their mathematical understandings through measurement contexts. This approach to teaching and learning mathematics was suggested by research in Russian education conducted by Davydov (1975), Minskaya (1975), and others.

Measurement is often thought of as the determination of size, amount, or degree of something by using an instrument or device marked in standard units. Measure Up, however, uses a broader definition of measurement that includes (1) comparing something with an object of known size; (2) estimating or assessing the extent, quality, value or effect of something; and (3) judging something by comparing it with a certain standard. These three aspects of measurement allow students to explore mathematical structures and develop an understanding of quantitative relationships that offers access to more sophisticated mathematical ideas at earlier grades.

This article focuses on one aspect of the first-grade Measure Up curriculum (Dougherty et al. 2004) related to students' understanding of unit. We describe some of the critical tasks that develop the concept of unit and summarize the effect on student development of a later concept of fractions.

Comparing Quantities to Develop the Concept of a Unit

Students begin first grade by identifying attributes of objects that can be compared. These attributes include size, shape, and color. Of all the attributes, size is the one that students use in the most general sense. They often say that this object is larger or smaller than that one, but they do not specify *how* it is larger or smaller.

To better describe size, students identify four ways that size can be measured or compared. These measures are length, area, volume, and mass. For each of these measures, students determine methods to compare two or more objects, called direct comparisons. In direct comparison (**fig. 1a–d**), they physically compare the quantities by placing them side by side to measure length; putting them on a balance scale to measure mass; using congruent containers to measure the volume of liquid or dry quantities; or placing one quantity on top of the other to measure area.

The need to label each quantity arises as

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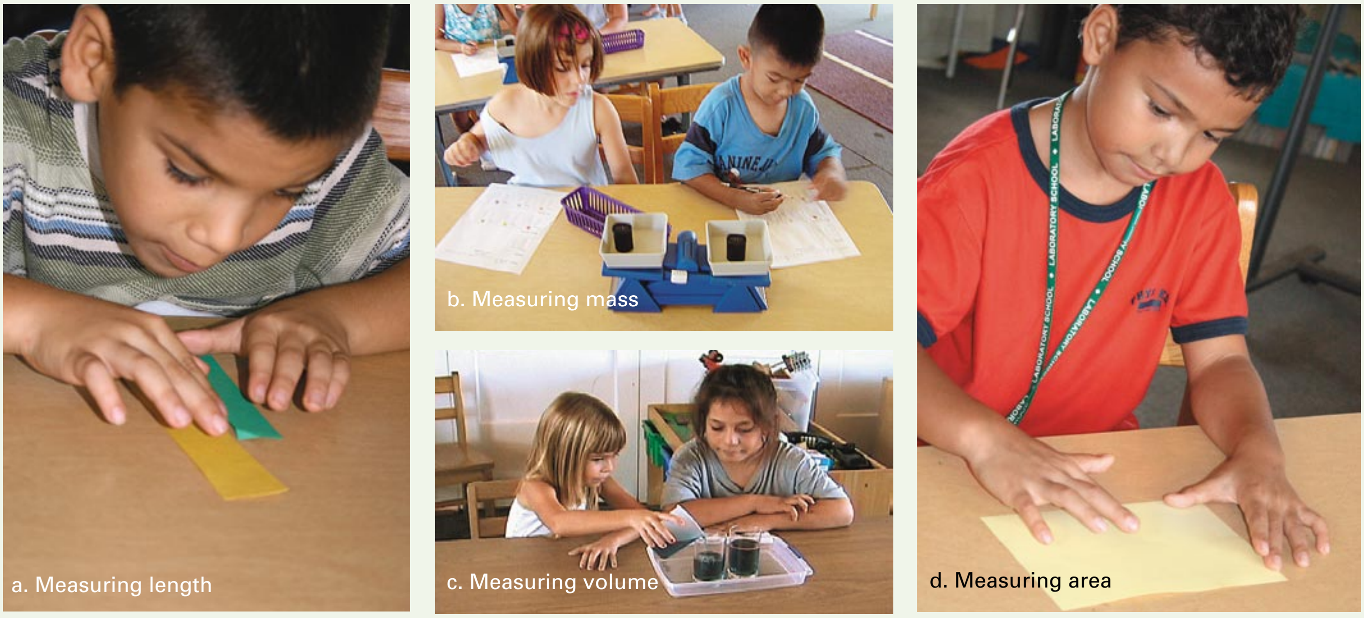
Figure 1

Students use direct comparisons to measure length, mass, volume, and area.



students attempt to communicate their findings from the direct comparisons. By selecting a different letter to represent each quantity, the students are able to show their comparisons. At first they use concrete models, a process that leads, in quick and smooth progression, to using symbolic statements one would typically find in a formal algebra program (e.g., $Y > E$, $D = K$).

After the students compare various quantities by directly measuring one against another, they are confronted with a problem similar to the one given by the teacher in **figure 2**. The students suggest ways they can compare the lengths of the various line segments without moving them. One way is to create a unit equal in length to length R. In essence, this unit represents the standard (or intermediate measure) by which the students can measure the other lengths. As they make comparisons to this intermediate measure, they generate statements about the comparisons. For example, the unit created is called length Q, and students can write $R = Q$ (read as "length R is equal to length Q"). As they measure, they might generate other statements such as $Q = H$, $L < Q$, and $B > Q$. From these statements, the students decide that length R is the same as length H, is longer than length L, and is shorter than length B. Besides developing deductive reasoning skills by creating and using a standard for measurement, the students are developing conceptual understandings about the symmetric and transitive properties of equality and inequality.

For example, the students recorded $R = Q$ and $Q > C$. Dustin described this relationship by saying, "If lengths R and Q are the same and length Q is more than length C, then R has to be more than C because R and Q are the same. You could write R greater than C" (authors' observation notes 2002). By physically measuring and recording the comparisons, the students are noting relationships

Figure 2

Sample problem for introducing intermediate units to first graders

When I got to school this morning, I found this note from our friend, Rabbert. Let me read you what he wrote:

Lani has promised to get me a special treat if I can figure out a way to compare the lengths of these line segments to length R. See, I put them up all around your room. I thought that first graders might be able to help me. I hope so because I love getting special treats from Lani! Signed, Rabbert.

What could we do? We can't take the line segments off the wall!

among and across multiple quantities without directly measuring them.

Using a Unit to Quantify Continuous Quantities

The question of how much larger or smaller the measure of one object is compared with the measure of another object arises quite naturally. While justifying why quantities are not equal, the students articulate what it is about their methods of comparison that reveals the difference between the quantities in question. Although they can easily point to the difference on a physical model, they have to decide how to describe the difference in words and symbols. Early on, they describe the differences by saying that, for example, volume W is greater than volume T by volume L. In this case, volume L is the difference (the amount by which volume W is greater than volume T) between volume W and volume T. The physical model can be used to illustrate volumes T, L, W, $W - L$, and $T + L$. To describe more precisely how much larger or smaller one quantity is compared with another, the students discuss the need to quantify volume L.

This recognition of the need to describe in words and symbols the difference in quantities establishes the need to introduce a unit to compose or decompose a quantity in order to quantify amounts. The students first begin with tasks that require them to create a quantity equal to another quantity that must be measured with a unit. The teacher may ask the students to complete a task similar to the one in **figure 3**.

After some discussion of what is and is not possible and appropriate, the students suggest using another (smaller) container to find a volume unit that will measure volume W a whole number of times and then using that same volume unit an equal number of times to create another volume equal to volume W. To keep track of the number of times they use the volume unit to measure volume W, the students record tally marks. This notation symbolizes the action that the students are doing. For example, if four volume units of E are needed to measure volume W, students would represent this comparison in this way:

$$E \xrightarrow{||||} W$$

This notation reads as volume unit E is used four times to make volume W. Volume W can be replicated if volume unit E is used the specified times. Students represent that relationship by using other mathematical statements, including these: $E + E + E + E = W$, $W = 3E + E$, $W = 4E$, and $W - 4E = 0$.

Through these tasks, first graders can decompose and then compose quantities in length, area, volume, and mass that use the iterations of a unit. As one first grader, Jason, described it, "You have to know the unit to measure something. And you have to remember how many times you used it to measure. And you have to be careful when you measure 'cause if you're measuring length, you can't leave any spaces or put a unit on top of a unit or you won't be able to know how many units it took to measure it" (authors' observation notes 2003).

The introduction of units allows students to move from direct comparisons to indirect comparisons. For example, they no longer need to physically place two lengths next to one another to compare them. They can use units to measure both and then compare the number of units used.

The students are given the quantities of two areas and then asked to decide how the two areas compare without cutting them out. Macy, a first grader, found an area unit that could be used to measure both quantities (see **fig. 4**). She used an area unit that looked like this rectangle and found

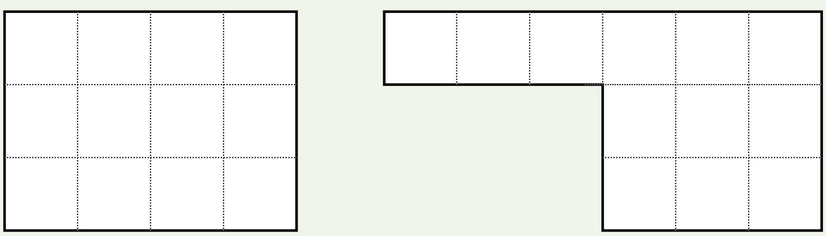
Figure 3

Sample problem for introducing units to first graders

We need to pour a volume equal to volume W into the empty container across the room. Volume W cannot be moved, and we don't have any other containers large enough to hold an equal volume. How could we create this new volume?

Figure 4

Macy's area quantities to compare



Area P Area R

that it took four iterations of the unit for both areas. She wrote these statements:

$P = 4E$
$\underline{4E = R}$
$P = R$

To describe her method, Macy said, "I made area unit *E* because it could fill up area *P* four times and it filled up area *R* four times. That's why I know the two areas are the same quantity because it took the same number of area unit *E*s to make them."

Macy was asked if there were other area units that could be used to measure the two areas. She thought there could be and proceeded to find one, which she later described. Macy found that multiple units could be used; although they varied in shape, both areas required the same number of units to measure them. She wrote the statements in **figure 5** to show the measure of the two areas with the different area units.

Macy made another discovery: "I can tell which area-units are bigger than others. If it takes more area-units to measure, then the unit is smaller. If it takes lesser area-units to measure, then the unit is bigger. The bigger the unit, the lesser times you have to use it. The smaller the unit, the more times you have to use it" (interview notes 2002).

Macy's description of the relationship among the unit, the number of iterations, and the quantity allows her to use other notations to symbolize it. Students write $P/E = 4$ (read "quantity *P* measured by *E* is 4") or $P/H = 6$. From just the symbolic statements, the students recognize that the quantity being measured is the same but that the units used to measure the quantity are different. By looking at the number of times the unit is used, students can decide that unit *E* must be larger than unit *H* because it took 4 unit *E*s to measure the quantity. As Reed (interview notes 2002) described it, "The bigger the unit, the more space it takes up, so it takes less of them to measure the quantity."

Figure 5

Macy's statements of measure

$3B = R$	$6H = R$	$2W = P$	$12X = P$	$1A = P$
$3B = P$	$P = 6H$	$2W = R$	$12X = R$	$1A = R$
$P = R$	$P = R$	$P = R$	$P = R$	$P = R$

Figure 6

Fraction problem for fourth graders

Sammi said, "3/4 is always greater than 2/3." "I disagree," said Carly. "I think there are some cases where 2/3 will be greater than 3/4." Whom do you agree with? Why? Support your answer with details that may include a drawing or diagram. (Dougherty, Zenigami, and Okazaki 2005)

Impact on Mathematics Learning in Later Years

The use of continuous, nonspecified quantities, linked to specific measurement with units, introduces students to the concept of number in a rich way that promotes flexible numerical thinking. As the Measure Up students move to other grade levels, their experience with units allows them to investigate rational numbers, for example, from the measurement perspective as they build number sense and articulate magnitude.

Measure Up students in grade 4 use the notion of unit to work with a fraction by thinking of the fraction in two ways. It can represent the quantity, or the whole; or it can represent the amount used to measure the whole. In the latter case, the denominator of a fraction tells how many units are needed to measure the whole. Because of the students' previous experience in describing the relationships between and among units and quantities, comparing fractional quantities becomes more meaningful.

One task we give to fourth-grade students is shown in **figure 6**. Students' responses to this task focused on the size of the unit. They indicated that Sammi is correct if the whole for both quantities is the same and that Carly is correct if the whole is not the same for both quantities. They conclude that to compare fractional quantities one must assume that the whole quantity is the same for everything being compared.

Measure Up students in grade 5 continue their development of fractions through explorations of the concept of theoretical probability. In this model, a rectangular area unit is subdivided into parts that represent occurring events. The fraction of the area represented by each part is the probability that a given event might occur.

Students are given an area unit created from colored square tiles and asked to compare the chances

Reflect and Discuss: *Measure Up for Understanding*

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether what you do works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "Measure Up for Understanding," by Barbara J. Dougherty and Linda C. H. Venenciano, are suggested prompts to aid you in reflecting on the article and on how the authors' ideas might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- **What mathematical understanding of measurement and algebra must a teacher have in order to implement these concepts effectively in classroom instruction?**
- **How can students benefit from thinking first about generalized concepts and later about specific numerical examples?**
- **How does learning mathematics from a measurement perspective influence a child's understanding of numeric relationships?**
- **What expectations of student behaviors must be communicated to build a classroom environment where students construct their understanding and explain their thinking?**

You are invited to tell us how you used "Reflect and Discuss" as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at tcm@nctm.org. Please include "Readers' Exchange" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgement. Letters are also edited for style and content.*

of randomly pulling a specific colored square tile. To decide, students used the fact that red tiles make up more of the area unit than do green tiles; thus, one has a greater chance of pulling a red tile than pulling a green tile.

Summary

In its preliminary results, Measure Up's approach to measurement (Davydov 1975) as the basis for mathematical development has shown that students proficiently use the relationships among and between units to compare quantities through indirect measures. We feel that by beginning with measurement we give students opportunities to concretely represent mathematical structures. This experience influences students' approach to new and more sophisticated mathematics topics. They become accustomed to pursuing a conceptual understanding that is well connected and logically fits into their theoretical framework of measurement.

As we move further into the Measure Up project, we are excited to see the longitudinal effects of using measurement as the basis for mathematics topics as the complexity of the mathematics increases. We hope that these early experiences will empower all students to confidently approach new topics and pursue higher-level mathematics at an earlier age and with greater understanding.

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